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Topological condensed matter: mathematical analysis of fully symmetric chains

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Introduction

Physics of mathematics and mathematics of physics

The discovery of topological phases of matter has reshaped our understanding of Condensed Matter Physics systems, revealing that quantum phases can be characterized by global, topological properties rather than local order parameters. In some sense this is the substantial difference between the perturbative approaches in physics, mostly local, and the non-perturbative approaches that make use of a more modern mathematics related to differential topology and algebraic topology. The undisputed usefulness of formal and abstract branches of pure mathematics in describing and organizing physical phenomena, emerging in the study of the topological phases of matter, should be taken as a lesson to reformulate all physical theory in a more modern, geometric and interconnected theoretical framework with the abstraction of pure mathematics. The feeling that "physics is first of all an empirical science" is something that could have been true perhaps 100 years ago; it is time to modernize and realize that the themes of physics have changed: theoretical physics, if it wants to survive as such, must open itself to the most modern mathematics.

In this thesis we aim to study the mathematical structure of Condensed Matter Physics systems with particular attention to topological insulators. Topological insulators are materials that behave as electrical insulators in their bulk while supporting conducting states at their boundaries. These edge or surface states are protected by topological invariants and are robust against perturbations that preserve the system's symmetries. The quantum Hall effect (QHE), discovered in 1980, is the first experimentally observed topological phase of matter. The crucial point is that the Hall conductance becomes quantized and corresponds to a topological invariant: the first Chern number of the Bloch bundle. The QHE thus provided the first physical realization of a topological insulator, where gapless edge modes coexist with an insulating bulk. This foundational example paved the way for the modern theory of topological insulators and superconductors.

The mathematics behind these physical system covers many areas: functional analysis with spectral theory, group and representation theory, vector and principal bundle theory with a particular focus on parallel transports and K-theory. The classification of topological insulators with the presence of discrete symmetries such as time reversal, charge conjugation and chiral symmetry through K-theoretic approach has led to the so-called ten-fold way. Each combination of discrete symmetries defines one of ten symmetry classes, known as the Altland-Zirnbauer classes. These classes serve as the foundation for the periodic table of topological insulators and superconductors, with each entry characterized by a K-theory group depending on spatial dimension.

In this framework, the Berry phase is a concept that reveals deep connections between quantum mechanics, geometry and topology. This is a geometric phase acquired by a quantum system when its parameters are varied adiabatically along a closed loop in parameter space. Unlike the dynamical phase, it depends only on the geometry of the path taken, not on the evolution time.

The main questions we address are the following:

> *Can the Berry phase be used to detect and classify topological phases? If so, how much topology does the Berry phase have access to? To what extent does the Berry phase invariant coincide with the invariants predicted by K-theory? Does the Berry phase record the presence of other structures?*

Chapter 1 introduces the mathematical foundations of quantum mechanics.
In Section 1.1 states are modeled as rays in a Hilbert space, while observables correspond to self-adjoint operators acting on it. Dynamics is described either by the Schrödinger equation or by the Heisenberg one, which are shown to be unitarily equivalent via Ehrenfest's theorem.
The spectral theory of unbounded self-adjoint operators is developed in detail in Section 1.2, culminating in two versions of the spectral theorem: the first using projection-valued measure and the second one using the concept of direct integral Hilbert space. These provide a rigorous framework for defining functions of operators and interpreting physical measurements in infinite-dimensional settings. A key technical result is Stone's theorem, which ensures that every strongly continuous one-parameter unitary group arises from the exponential of a self-adjoint generator. This underlies the mathematical structure of quantum dynamics and is central to the analysis of translation-invariant systems.
In Section 1.3 symmetries are discussed. Symmetries play a fundamental role: by Wigner's theorem, every symmetry that preserves transition probabilities is implemented by a unitary or anti-unitary operator. This leads to a group-theoretic classification of symmetries, which are shown to act by conjugation on observables. We also introduce the notion of dynamical symmetries, explore the Lie group structure of symmetry groups and the related notion of Lie algebras. The Chapter ends with a discussion about projective unitary representation and their link with group cohomology.
Appendices A and B cover, respectively, a complete proof of Wigner's Theorem and the introduction to the Gelfand-Naimark-Segal (GNS) construction, which builds representations of C^*-algebras from positive linear functionals. This abstract framework allows for inequivalent quantizations of the same physical system, with applications ranging from algebraic quantum field theory to polymer quantization and big bounce theory.

Chapter 2 connects Quantum Mechanics, specifically Condensed Matter Physics systems, to bundle theory.
In Section 2.2, we analyze periodic systems using the Zak-Bloch-Floquet transform which allows the decomposition of the Hamiltonian into fibered operators over the Brillouin torus. This gives rise to the Bloch bundle, a vector bundle whose fibers consist of images of the spectral eigenprojectors of the fibered Hamiltonians.
The geometry and topology of this bundle encode essential physical information also linked to the possibility of quantized responses. To study these bundles rigorously, we introduce the theory of vector bundles and connections on them in Section 2.3.
Appendices C, D and E cover, respectively, results on the horizontal lift and the parallel transport a detailed presentation of curvature and Chern classes and the classification of vector bundles up to isomorphism.

Chapter 3 focuses on the classification of topological phases of quantum matter

using K-theory.
In Section 3.1 we discuss the definition and some properties of K-theory.
From the point of the symmetry operators, the main actors are time reversal T, particle hole C and chiral S symmetries introduced in Section 3.2.
Depending on their presence and mutual relations, quantum systems fall into one of ten symmetry classes, the so-called Altland-Zirnbauer classes, which are organized in the ten-fold way presented in some detail in Section 3.3. However, the K-theory classification is highly abstract: it provides existence and uniqueness results but does not give explicit formulas for topological invariants. In contrast, the Berry phase and Berry curvature yield computable, physically meaningful quantities. This raises the central questions of the thesis.

Chapter 4 is the core of this thesis containing some **original results**: it addresses the main questions in the context of fully symmetric BDI, DIII, CI and CII chains with finitely many degrees of freedom, say $2m$. These are gapped quantum systems with strong symmetry constraints and serve as a testing ground for comparing different types of topological invariants.
We begin by explicitly constructing the action of symmetry operators on fibered Hamiltonians in Section 4.1 and then build symmetric Bloch bases using transport parallel tools in Section 4.2.
This allows us to define the Berry phase, computed along the Brillouin torus, in Section 4.3. In the same Section we show, using the winding number for unitary matrices, that the Berry phase transforms in a well defined way under a change of symmetric Bloch bases. Taking the appropriate quotient we construct a $\mathbb{Z}_2$ gauge invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ that we show to be a topological invariant. This invariant turns out to be independent from the specific symmetry class providing a topological invariant for all symmetry classes. However some classes have no non-trivial topological phases and so we need to take into consideration the specific structure given by the squaring to $\pm 1_{\mathbb{C}^{2m}}$ of the discrete symmetry operators. The outcome is that in the presence of a quaternionic structure, i.e. an anti-linear operator squaring to $-1_{\mathbb{C}^{2m}}$, the invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ has to be vanishing.
In the end, in Section 4.4, we focus on the BDI symmetry class which is the only one, among the fully symmetric, not to have quaternionic structure. In this case, the ten-fold way labels the chain with a $\mathbb{Z}$ topological invariant while the Berry phase provides a $\mathbb{Z}_2$ topological invariant. Considering the quite general example of the Kitaev chain with arbitrary long-range couplings we show that the invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ could be interpreted as the parity of the full topological invariant.

In the end, this thesis lies at the intersection of Condensed Matter Physics, functional analysis and differential topology. By grounding physical intuition in precise mathematical structures we aim to clarify if and when geometric invariants, such as the Berry phase, reflect the full topological content of a quantum system and if it is sensitive to other mathematical structures.

Contents

Chapter 1

Quantum Mechanics primer

The modern description of matter inevitably passes through quantum theory. The introduction and development of Quantum Mechanics had its roots in the range 1900-1920 when physics faced the so-called "crisis of classical physics". The key point was that some physical phenomena escaped the description of classical Newtonian and Maxwellian physics, so much so that a new formulation of physics had to be developed in terms of more modern mathematics. This new physical theory, which has brought about an epistemological and/or philosophical revolution in our understanding of reality, has been applied with great success (both theoretical and empirical) at the molecular and atomic scale giving rise to the modern understanding of atomic spectra, molecular dynamics and of the periodic table. Today, Quantum Mechanics is the basis of modern information, computation and communication technology driving the modern technological/energy transition of our world. As far as we are concerned, we are interested in quantum theory because it is the basis of our understanding of solid state physics and specifically plays a fundamental role in the topological phases of matter we will be dealing with.

In Section 1.1 we discuss the basic elements of a quantum theory: states and observables. We define what a physical state is and what kind of operators are good quantum observables; we also introduce the differential equations that describe the dynamics of states and observables.

In Section 1.2 we introduce the spectral theory for unbounded self-adjoint operators motivated by the fact that self-adjoint operators have real spectrum and so they are good quantum observables according to Quantum Mechanics axioms. We then discuss the various forms of the spectral theorem introducing concepts like projection-valued measures, functional calculus and the direct integral Hilbert spaces. These concepts are fundamental to having a good understanding of Quantum Mechanics to avoid the inappropriate use of terms, notions and constructions or to avoid an oversimplification of the topics which can lead to a debasement of the physical theory itself. Furthermore, thanks to these concepts, we discuss Stone's Theorem, we solve the dynamical equation in the case of a time-independent Hamiltonian and we prove Ehrenfest's Theorem that states the unitarily equivalence between the Heisenberg and the Schrödinger pictures. In order to have the possibility to define more general, i.e. non-unitarily equivalent, pictures of Quantum Mechanics, we briefly discuss the Gelfand-Naimark-Segal construction in Appendix A.

In Section 1.3 we study the symmetries in a quantum theory. We first define the quantity we want to be preserved by a symmetry, both from the physical point of view, the transition probability, and from the mathematical point of view, the pseudo-angle. We prove Wigner's theorem in Appendix B, we discuss how symmetries are implemented at the level of observables on the Hilbert space of the quantum system and the relations between symmetries and group cohomology.

The main references for this Chapter are [1, 2, 3, 4, 5, 6, 7].

1.1 Basics: states and observables

In Quantum Mechanics, as in every physical theory, the characterization of states and observables is the starting point. The former describes properties of the physical system at fixed time, while the latter are quantities that we can measure in experiments and through which we can extract information about the system. Moreover, there are laws that dictate how systems evolve and interact over the time and give the dynamics of the system.

In any quantum mechanical system, particles are assumed to move in a configuration space X, which is assumed to be a measurable space (X, Ω) and so a measure space once we endow it with measure dx. In particular, this allows us to define the Hilbert space $\mathcal{H} := L^2(X, \Omega, dx)$; the state of a particle is then described by a so-called ray in $\mathcal{H}$, that is an element of the projective Hilbert space $\mathbb{P}\mathcal{H}$. Two vectors that differ by a non-zero scalar factor correspond to the same physical quantum state represented by a ray in Hilbert space, which is an equivalence class in $\mathcal{H}$ and, under the natural projection map $\pi : \mathcal{H} \to \mathbb{P}\mathcal{H}$, an element of $\mathbb{P}\mathcal{H}$. Hence

$$\mathbb{P}\mathcal{H} := \{[\psi] \mid \psi = \lambda\psi' \text{ with } \psi, \psi' \in \mathcal{H} \text{ and } \lambda \in \mathbb{C}^*\} \equiv \mathcal{H} \setminus \{0\}/{\sim_1}\,, \qquad (1.1)$$

where $\psi \sim_1 \psi'$ if $\psi = \lambda\psi'$ with $\lambda \in \mathbb{C}^*$. An alternative, equivalent description of the state space of a quantum mechanical particle, which is often preferred, is given in terms of the sphere, $\mathbb{S}(\mathcal{H})$, in the Hilbert space $\mathcal{H}$ as

$$\mathbb{P}\mathcal{H} := \{[\psi] \mid \psi = e^{i\theta}\psi' \text{ with } \psi, \psi' \in \mathbb{S}(\mathcal{H}) \text{ and } \theta \in \mathbb{R}\} \equiv \mathbb{S}(\mathcal{H})/{\sim_2}\,, \qquad (1.2)$$

where $\psi \sim_2 \psi'$ if $\psi' = e^{i\theta}\psi$ with $\theta \in \mathbb{R}$. In this sense phases are irrelevant in Quantum Mechanics: assuming that $|\psi(x)|^2 dx$ is the probability density of finding the quantum particle in an infinitesimal region around $x \in X$, the identification $\psi' = e^{i\theta}\psi$ with $e^{i\theta} \in \mathrm{U}(1)$ preserves this probability density.
The two definitions are in fact equivalent and one can pass from one to the other by a normalization procedure of vectors; therefore we have the following Definition.

Definition 1.1.1 (Physical state)**.** *A physical state is an element of the projective Hilbert space $\mathbb{P}\mathcal{H}$.*

We underline that physical states can be formulated in terms of observables, rather than as vectors in a vector space. These are positive normalized linear functionals on a C*-algebra, or sometimes other classes of algebras of observables. This kind of construction, called Gelfand-Naimark-Segal construction [8], is fundamental to formulate unitarily inequivalent formulation of Quantum Mechanics like Polymer

Quantum Mechanics [9, 10]. C*-algebras and the Gelfand-Naimark-Segal construction are going to be discussed in Appendix A.

In the realm of Quantum Mechanics, observables are given by linear self-adjoint operators on the Hilbert space $\mathcal{H}$. Let us assume that the operator $O : \mathcal{H} \to \mathcal{H}$ is bounded, i.e. there is a constant C such that $||O\psi|| \le C||\psi||$ for all $\psi \in \mathcal{H}$, where $||\cdot||$ is the norm in the Hilbert space $\mathcal{H}$ that is induced by the hermitian product $\langle \cdot, \cdot \rangle$. We define the adjoint of O.

Definition 1.1.2 (Adjoint)**.** *Suppose $\mathcal{H}$ is a complex Hilbert space, with hermitian product $\langle \cdot, \cdot \rangle$. Consider a bounded linear operator $O : \mathcal{H} \to \mathcal{H}$. Then the adjoint of O is the bounded linear operator $O^* : \mathcal{H} \to \mathcal{H}$ satisfying*

$$\langle O\psi, \phi \rangle = \langle \psi, O^*\phi \rangle \qquad \forall\ \psi, \phi \in \mathcal{H}.$$

Existence and uniqueness of this operator follows directly from the Riesz representation theorem: indeed if O is bounded there exists a unique element ϕ' such that

$$\langle O\psi, \phi \rangle = \langle \psi, \phi' \rangle \qquad \forall\ \psi \in \mathcal{H} \tag{1.3}$$

and we define O^* as the only operator such that $O^*\phi = \phi'$. The set of bounded linear operators on a complex Hilbert space $\mathcal{H}$ together with the adjoint operation and the operator norm form the prototype of a C*-algebra denoted as $\mathcal{B}(\mathcal{H})$.
However, in Quantum Mechanics most of the operators are unbounded, like the momentum operator, and the Riesz representation theorem loses its validity. In this case, it is possible to define the adjoint operator of densely defined operators, that is, operators such that the closure of the domain coincides with the entire vector Hilbert space. Let us start with the following Definition.

Definition 1.1.3 (Densely defined linear operator)**.** *A densely defined linear operator O from one Hilbert space X to another one Y is a linear operator that is defined on a dense linear subspace* $\mathrm{Dom}(O)$ *of X and takes values in Y.*

The same definition can be given in the more general case of topological vector spaces. On densely defined linear operator we can define the adjoint using again the Riesz representation theorem on the appropriate domains $\mathrm{Dom}(O)$ and $\mathrm{Dom}(O^*)$.

The probabilistic interpretation of Quantum Mechanics leads to being able to define only statistical information by measuring an observable $O = O^*$, i.e. self-adjoint operators. In particular, its expectation value on the state is postulated to be

$$\langle O \rangle_{[\psi]} := \langle \psi | O\psi \rangle, \tag{1.4}$$

where the notation $\langle \psi | \phi \rangle := \langle \phi, \psi \rangle$ is the standard physics notation. Definition (1.4) does not depend on the choice of a representative ψ of the physical state; moreover, the real-valuedness of these expectation values follows from the fact that the spectrum of a self-adjoint operator is real as we will see.

Another important piece of information is the dynamics of a quantum system. From now on we fix natural units $c = \hbar = G = 1$. To specify the dynamics of a quantum system one first needs to select the observable $H = H^*$ which corresponds to the energy of the system: this particular operator is the Hamiltonian. We can then equivalently prescribes the dynamical evolution of the state of the system or of the observables. On the one hand, the time evolution of a quantum state $[\psi] \in \mathbb{P}\mathcal{H}$ is dictated by the Schrödinger equation that reads

$$i\partial_t \psi(t) = H\psi(t), \qquad \psi(t=0) = \psi_0 \in \mathcal{H}, \tag{1.5}$$

that gives us a deterministic evolution of a probabilistic information. On the other hand, the evolution of observables is dictated by the Heisenberg equation

$$i\partial_t O(t) = [O(t), H], \qquad O(t=0) = O_0 = O_0^*. \tag{1.6}$$

In Paragraph 1.2 we are going to solve these equations and to prove the Ehrenfest theorem that makes a deep connection, i.e. the equivalence, between these two pictures.

1.2 Basic spectral theory

In Quantum Mechanics we often deal with infinite dimensional spaces and unbounded operators. Here we give a brief introduction to the difficulties that can be encountered and to the spectral theory that helps us to solve them.

Suppose $\mathcal{H}$ is a finite dimensional Hilbert space and O is a self-adjoint linear operator on $\mathcal{H}$, i.e. $\langle \phi | O\psi \rangle = \langle O\phi | \psi \rangle$ for all $\phi, \psi \in \mathcal{H}$; then there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors for O with real eigenvalues. Since there is a standard notion of orthonormal bases for general Hilbert spaces, we might hope that a similar result would hold for self-adjoint operators on infinite-dimensional Hilbert spaces. However, already in the simple case of $\mathcal{H} := L^2([0,1])$ and an operator O on $\mathcal{H}$ defined by

$$(O\psi)(x) = x\psi(x),$$

we see that the operator is self-adjoint since $\langle \phi | O\psi \rangle = \langle O\phi | \psi \rangle$ but it has no eigenvectors since if

$$(O\psi)(x) = x\psi(x) = \lambda\psi(x),$$

then ψ would have to be supported on the set $E := \{x \in [0,1] \mid x = \lambda\}$ which is a set of measure zero. Physicists would say that the operator O does have eigenvectors, namely the distributions $\delta(x-\lambda)$: they indeed satisfy $x\delta(x-\lambda) = \lambda\delta(x-\lambda)$, but they do not belong to the Hilbert space. Such "eigenvectors", which belong to some larger space than $\mathcal{H}$, are known as generalized eigenvectors: even though they are not actually in the Hilbert space, we may hope that there is some sense in which they form something like an orthonormal basis.
Moreover, there is another serious difficulty that arises with self-adjoint operators in the infinite-dimensional case. Most of the self-adjoint operators O of Quantum Mechanics are unbounded operators. The crucial point is that the Hellinger-Toeplitz theorem, which is an application of the closed graph theorem, states that an everywhere defined symmetric (hence self-adjoint) operator on a Hilbert space is necessarily bounded. Thus, if O is not bounded and self-adjoint, it cannot be defined on the whole Hilbert space. Therefore we can define unbounded self-adjoint operators only on a dense subspace of the whole Hilbert space according to Definition 1.1.3.

One of the major problems in dealing with unbounded operators is the impossibility of defining functions of operators via power series due to convergence issues. The problem is solved thanks to the spectral theorem which provides us with a functional calculus for self-adjoint unbounded operators; furthermore, the spectral theorem provides a probability distribution for the result of measuring a self-adjoint operator. For an operator which has a true orthonormal basis of eigenvectors $\{e_j\}$ with corresponding eigenvalues $\{\lambda_j\}$ the probability is built as follows: given any Borel set[1] $E \subset \mathbb{R}$, let V_E be the closed span of all the eigenvectors for O with eigenvalues in E and let P_E be the orthogonal projection onto V_E. Then for any unit vector $\psi \in \mathcal{H}$ we have

$$\mathrm{Prob}_\psi(\lambda_j \in E) = \langle \psi | P_E \psi \rangle.$$

In particular, if the eigenvalues are distinct and ψ decomposes as $\psi = \sum_j c_j e_j$, the probability of observing the value λ_j will be $|c_j|^2$ since P_{λ_j} is just the projection onto e_j.
In cases where O does not have a true orthonormal basis of eigenvectors, we would like the spectral theorem to provide a family of projection operators P_E, one for each Borel subset $E \subset \mathbb{R}$, which will allow us to define probabilities as before. We will call these projection operators "spectral eigenprojections" and the associated subspaces V_E "spectral subspaces". Thus, P_E is the orthogonal projection onto V_E and, intuitively, V_E may be thought as the closed span of all the generalized eigenvectors with eigenvalues in E.
In the first version of the spectral theorem both these goals will be achieved, with the spectral eigenprojections being provided by a projection-valued measure and the functional calculus being provided by integration with respect to this measure. Although having (generalized) eigenvectors for a self-adjoint operator is, from a practical standpoint, of secondary importance, we provide a framework for understanding such eigenvectors, using the concept of a direct integral. The second version of the spectral theorem decomposes the Hilbert space as a direct integral, with respect to a certain measure μ, of generalized eigenspaces for a self-adjoint operator O. The generalized eigenspace for a particular eigenvalue λ will not actually be a non-trivial subspace of the Hilbert space, unless $\mu(\{\lambda\}) > 0$. Thus, the notion of a direct integral gives a rigorous meaning to the notion of “eigenvectors” that are not actually in the Hilbert space.

1.2.1 Unbounded self-adjoint operators on a Hilbert space

The definition of unbounded operator O is essentially the one given in Definition 1.1.3. In the case of an unbounded operator O the definition of the adjoint is more delicate.

Definition 1.2.1 (Adjoint)**.** *Suppose O is an operator defined on a dense subspace* $\mathrm{Dom}(O) \subset \mathcal{H}$*. Let* $\mathrm{Dom}(O^*)$ *to be the space of all $\phi \in \mathcal{H}$ for which the linear functional*

$$\Phi : \psi \mapsto \langle O\psi | \phi \rangle \qquad \forall\ \psi \in \mathrm{Dom}(O)$$

[1]Recall that a Borel set is any subset of a topological space that can be formed from its open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement.

is bounded. For $\phi \in \mathrm{Dom}(O^*)$*, we define* $O^*\phi$ *to be the unique vector such that*

$$\langle O\psi|\phi\rangle = \langle\psi|O^*\phi\rangle \qquad \forall\ \psi \in \mathrm{Dom}(O).$$

In the bounded case, the linear functional Φ is bounded for every $\psi \in \mathcal{H}$ and Definition 1.2.1 reduces to Definition 1.1.2. We can now define when an unbounded operator is self-adjoint.

Definition 1.2.2 (Self-adjoint unbounded operator)**.** *An unbounded operator* O *on* $\mathcal{H}$ *is self-adjoint if*

$$\mathrm{Dom}(O) = \mathrm{Dom}(O^*) \qquad \text{and} \qquad O\phi = O^*\phi \qquad \forall\ \phi \in \mathrm{Dom}(O).$$

We may reformulate the definition of self-adjointness by saying that O is self-adjoint if O^* is equal to O, provided that equality of unbounded operators is understood to include equality of domains. In cases where the unbounded operator O is not self-adjoint, there is no need to despair: there is still the possibility that it is essentially self-adjoint. This means that there is a unique extension of O, given by its closure[2], which is self-adjoint.

We now define the spectrum of an unbounded operator.

Definition 1.2.3 (Spectrum and resolvent)**.** *Suppose* O *is an unbounded operator on* $\mathcal{H}$*. A number* $\lambda \in \mathbb{C}$ *belongs to the resolvent set of* O*, called* $\rho(O)$*, if there exists a bounded operator* B *with the following properties:*

1. $B\psi \in \mathrm{Dom}(O) \qquad \forall\ \psi \in \mathcal{H}$;
2. $(O - \lambda 1_{\mathcal{H}})B\psi = \psi \qquad \forall\ \psi \in \mathcal{H}$;
3. $B(O - \lambda 1_{\mathcal{H}})\psi = \psi \qquad \forall\ \psi \in \mathrm{Dom}(O)$.

If no such bounded operator B *exists, then* λ *belongs to the spectrum of* O*, called* $\sigma(O)$*.*

Note that even if O is self-adjoint, points $\lambda \in \sigma(O)$ are not necessarily eigenvalues; that is, there does not necessarily exist a non-zero $\psi \in \mathrm{Dom}(O)$ with $O\psi = \lambda\psi$. On the other hand, if $O\psi = \lambda\psi$ for some $\psi \in \mathrm{Dom}(O)$, then $O - \lambda 1_{\mathcal{H}}$ is not injective and thus λ certainly does belong to the spectrum of O. The spectrum of a self-adjoint operator has important properties as stated by the following Proposition.

Proposition 1.2.1 (Spectrum of a self-adjoint unbounded operator)**.** *If* O *is an unbounded self-adjoint operator on* $\mathcal{H}$*, the spectrum of* O *is contained in the real line.*

Proof. Consider a complex number $\lambda = a + ib$ with $b \neq 0$; we can compute that

$$\langle (O - \lambda 1_{\mathcal{H}})\psi | (O - \lambda 1_{\mathcal{H}})\psi\rangle \geq b^2 \langle\psi|\psi\rangle \qquad \forall\ \psi \in \mathrm{Dom}(O);$$

that shows that $O - \lambda 1_{\mathcal{H}}$ is injective. Moreover, we have

$$(\mathrm{Range}(O - \lambda 1_{\mathcal{H}}))^{\perp} = \mathrm{Ker}((O - \lambda 1_{\mathcal{H}})^*) = \mathrm{Ker}(O - \bar{\lambda} 1_{\mathcal{H}})$$

[2]Given an unbounded operator O its closure is the operator that has graph given by the closure of the graph of O.

and since $\bar{\lambda}$ has non-zero imaginary part we get that $O - \bar{\lambda}1_{\mathcal{H}}$ is injective and $(\text{Range}(O - \lambda 1_{\mathcal{H}}))^{\perp} = \{0\}$. Therefore, $\text{Range}(O - \lambda 1_{\mathcal{H}})$ is dense in $\mathcal{H}$. Since a self-adjoint operator is closed, it turns out that also $\text{Range}(O - \lambda 1_{\mathcal{H}})$ is closed and therefore we have $\text{Range}(O - \lambda 1_{\mathcal{H}}) = \mathcal{H}$. We have shown, then, that $O - \lambda 1_{\mathcal{H}}$ maps $\text{Dom}(O)$ injectively onto $\mathcal{H}$. It follows from the closed graph theorem that the inverse operator is bounded, so that $\lambda \in \rho(O)$. This shows that $\lambda \in \sigma(O)$ only if $\lambda \in \mathbb{R}$. □

Under a more accurate analysis, it is possible to show that $\sigma(O)$ is a closed subset of the real line and it can be shown (using the spectral theorem) that if a self-adjoint operator has bounded spectrum, then the operator must be bounded. We end with the following Definition and Proposition.

Definition 1.2.4 (Bounded from below)**.** *Let O be an unbounded operator on $\mathcal{H}$. Then O is bounded from below by $c \in \mathbb{R}$ if*

$$\langle O\psi|\psi\rangle \geq c||\psi||^2 \qquad \forall\ \psi \in \text{Dom}(O).$$

We say that O is non-negative if $c = 0$.

A useful result linked to the spectral gap hypothesis we are going to introduce later on, is the following one.

Proposition 1.2.2 (Spectral gap)**.** *Let O be an unbounded self-adjoint operator on $\mathcal{H}$. If O is bounded from below by c, then $\sigma(O) \in [c, +\infty)$.*

Proof. Suppose O is bounded from below by c and λ is a point in the spectrum of O. If ψ_n is a sequence of normalized vectors such that

$$\lim_{n\to+\infty} ||(O - \lambda 1_{\mathcal{H}})\psi_n|| = 0,$$

then we have, by Cauchy-Schwarz inequality, that

$$\lim_{n\to+\infty} |\langle (O - \lambda 1_{\mathcal{H}})\psi_n|\psi_n\rangle| \leq \lim_{n\to+\infty} ||(O - \lambda 1_{\mathcal{H}})\psi_n|| = 0.$$

Moreover, $O = \lambda 1_{\mathcal{H}} + (O - \lambda 1_{\mathcal{H}})$ so

$$\langle O\psi_n|\psi_n\rangle = \lambda + \langle (O - \lambda 1_{\mathcal{H}})\psi_n|\psi_n\rangle;$$

hence $\langle O\psi_n|\psi_n\rangle$ converge to λ as $n \to +\infty$. Since O is bounded below by c, we must have $\lambda \geq c$. □

1.2.2 Spectral theorem I

Given a self-adjoint operator O, we want to associate with each Borel set $E \subset \sigma(O)$ a closed subspace V_E of $\mathcal{H}$, where intuitively V_E is the closed span of the generalized eigenvectors for O with eigenvalues in E. We want that the collection of these subspaces have the following properties

1. $V_{\sigma(O)} = \mathcal{H}$ and $V_{\emptyset} = \{0\}$;

2. If E_1 and E_2 are disjoint, then $V_{E_1} \perp V_{E_2}$;
3. For any E_1 and E_2 we have $V_{E_1 \cap E_2} = V_{E_1} \cap V_{E_2}$;
4. if $\{E_j\}_{j \in J}$ are disjoint and $E = \cup_j E_j$, then $V_E = \oplus_j V_{E_j}$;
5. For any E, V_E is invariant under O;
6. if $E \subset [\lambda - \epsilon, \lambda + \epsilon]$ and $\psi \in V_E$, then $||(O - \lambda 1_{\mathcal{H}})\psi|| \leq \epsilon ||\psi||$.

A V_E so made is called spectral subspace. It is convenient to describe closed subspaces of a Hilbert space in terms of the associated orthogonal projection operators. Recall that, given a closed subspace V of $\mathcal{H}$, there exists a unique bounded operator P that equals the identity on V and equals zero on the orthogonal complement of V. This operator is called the orthogonal projection onto V and satisfies $P^2 = P = P^*$. Conversely, if P is any bounded operator on $\mathcal{H}$ satisfying $P^2 = P = P^*$, then P is the orthogonal projection onto a closed subspace $V = \text{Range}(P)$. The following Definition expresses the first four properties of spectral subspaces in terms of the corresponding orthogonal projections.

Definition 1.2.5 (Projection-valued measure)**.** *Let X be a set and Ω a σ-algebra in X. A map $\mu : \Omega \to \mathcal{B}(\mathcal{H})$ is called a projection-valued measure if the following properties are satisfied:*

1. *$\mu(\emptyset) = 0$ and $\mu(X) = 1_{\mathcal{H}}$;*
2. *for each $E \in \Omega$, $\mu(E)$ is an orthogonal projection;*
3. *for all $E_1, E_2 \in \Omega$, we have $\mu(E_1 \cap E_2) = \mu(E_1)\mu(E_2)$;*
4. *if $\{E_j\}_{j \in J}$ are disjoint, then $\forall\ \psi \in \mathcal{H}$ we have $\mu(\cup_j E_j)\psi = \sum_j \mu(E_j)\psi$.*

The spectral subspace associated to $\mu(E)$ is simply given by

$$V_E := \text{Range}(\mu(E)). \tag{1.7}$$

We note that, for any projection-valued measure μ and $\psi \in \mathcal{H}$, we can build up an ordinary (positive) real-valued measure μ_ψ by setting

$$\mu_\psi(E) := \langle \mu_\psi(E)\psi | \psi \rangle \qquad \forall\ E \in \Omega; \tag{1.8}$$

by normalizing we get a probability measure. This observation provides a link between integration with respect to a projection-valued measure and integration with respect to an ordinary measure. Indeed there exists a unique linear map $\Phi : f \mapsto \int_\Omega f d\mu$ that maps every function in the space of measurable complex-valued, not necessarily bounded, functions on Ω to an unbounded operator such that

$$\left\langle \left(\int_X f d\mu \right) \psi \middle| \psi \right\rangle = \int_X f(\lambda) d\mu_\psi(\lambda) \qquad \forall\ \psi \in \mathcal{H}, \tag{1.9}$$

where the domain of the integral is

$$W_f := \left\{ \psi \in \mathcal{H} \;\middle|\; \int_X |f(\lambda)|^2 d\mu_\psi(\lambda) < +\infty \right\}. \tag{1.10}$$

We underline that the integral with respect to this projection-valued measure is multiplicative. We are now ready to state the spectral theorem in its first form.

Theorem 1.2.1 (Spectral theorem I)**.** *Let O be a self-adjoint operator on $\mathcal{H}$. Then there is a unique projection-valued measure $\mu^O : \sigma(O) \to \mathcal{B}(\mathcal{H})$ such that*

$$O = \int_{\sigma(O)} \lambda d\mu^O(\lambda).$$

Since the spectrum of O is typically an unbounded set, the function $f(\lambda) = \lambda$ is an unbounded function on $\sigma(O)$. Note also that the equality in the spectral theorem includes, as always in the case of unbounded operators, equality of domains. That is, the domain of the integral, i.e. $W_f = \{\psi \in \mathcal{H} \mid \int_{\sigma(O)} |\lambda|^2 d\mu_\psi^O(\lambda) < +\infty\}$, coincides with $\mathrm{Dom}(O)$.
The spectral theorem gives us the possibility to define functions of operator without requiring to define it by a power series; this method is called functional calculus.

Definition 1.2.6 (Functional calculus)**.** *For any measurable function f on $\sigma(O)$, define a (possibly unbounded) operator by*

$$f(O) = \int_{\sigma(O)} f(\lambda) d\mu^O(\lambda).$$

Since the integral with respect to μ^O is multiplicative, it follows from Definition 1.2.6 that if $f(\lambda) = \lambda^N$ for some positive integer N, then $f(O)$ is the Nth power of O. Further, if $f(\lambda)$ is given by a uniformly convergent series supported on a compact subset $\sigma(O)$, the operator $f(O)$, computed using the functional calculus for the function $f(\lambda)$, may be computed also as a power series.

We remark that everything we have said extends immediately from $\sigma(O)$ to the whole real line by extending to zero the projection-valued measure outside the spectrum of the operator under examination.

1.2.3 Spectral theorem II

Here the goal is to state the spectral theorem in terms of a direct integral. The set up is a measure space (X, Ω, μ) with μ a σ-finite measure on the σ-algebra Ω and a separable Hilbert space $\mathcal{H}_\lambda$ for each $\lambda \in X$ with a inner product $\langle \cdot, \cdot \rangle_\lambda$. We want to define the direct integral of the $\mathcal{H}_\lambda$ with respect to μ. Elements of the direct integral will be sections s, meaning that s is a function on X with values in the union of the $\mathcal{H}_\lambda$, having the property that

$$s(\lambda) \in \mathcal{H}_\lambda \qquad \forall\, \lambda \in X. \tag{1.11}$$

Effectively the direct integral is the space of sections of a fiber bundle (E, π, B) whose base is $B = X$ and whose fibers are $\mathcal{H}_\lambda$. Indeed, s is a map such that

$$s : X \to \{\lambda\} \times \mathcal{H}_\lambda \simeq \mathcal{H}_\lambda, \qquad \lambda \mapsto s_\lambda = (\lambda, s(\lambda))$$

for every $\lambda \in X$ and $\pi(s_\lambda) = \lambda$.
The last technicality that we have to discuss is the measurability issue: it does not make sense to ask whether a section s is measurable, since the space in which $s(\lambda)$ takes its values is different for each λ. Hence, we need to introduce some additional structure that gives rise to a notion of measurability, called measurability structure.

One way to address the measurability issue is to choose a simultaneous orthonormal basis for each of the Hilbert spaces $\mathcal{H}_\lambda$. To deal with the possibility that different spaces can have different dimensions, we allow some of the vectors in our basis to be zero with the non-zero vectors forming an orthonormal basis of $\mathcal{H}_\lambda$ in the usual sense. We now define a simultaneous orthonormal basis for a family $\{\mathcal{H}_\lambda\}_{\lambda\in X}$ of separable Hilbert spaces.

Definition 1.2.7 (Simultaneous orthonormal basis)**.** *Given a family $\{\mathcal{H}_\lambda\}_{\lambda\in X}$ of separable Hilbert spaces, a collection $\{e_j(\cdot)\}_{j=1}^{+\infty}$ of sections with the property that, for each λ, $\{e_j(\lambda)\}_{j=1}^{+\infty}$ is an orthonormal basis for $\mathcal{H}_\lambda$ is called a simultaneous orthonormal basis for $\{\mathcal{H}_\lambda\}_{\lambda\in X}$.*

Provided that the function $\Psi : \lambda \mapsto \dim(\mathcal{H}_\lambda)$ is a measurable function from X into $[0, +\infty]$, it is possible to choose a simultaneous orthonormal basis $\{e_j(\cdot)\}_{j=1}^{+\infty}$ such that $\langle e_j(\lambda), e_k(\lambda)\rangle$ is measurable for all j and k. Having chosen a simultaneous orthonormal basis with this property, we define a notion of measurability for a section s.

Definition 1.2.8 (Measurable section)**.** *Given a simultaneous orthonormal basis $\{e_j(\cdot)\}_{j=1}^{+\infty}$, a section s is defined to be measurable if the function*

$$\Phi : \lambda \mapsto \langle e_j(\lambda), s(\lambda)\rangle$$

is a measurable complex-valued function for each j.

The squared norm of a section s is defined by

$$||s||^2 := \int_X \langle s(\lambda), s(\lambda)\rangle_\lambda d\mu(\lambda) \tag{1.12}$$

and, more generally, the inner product of two sections s_1 and s_2 (with finite norm) by

$$\langle s_1, s_2\rangle := \int_X \langle s_1(\lambda), s_2(\lambda)\rangle_\lambda d\mu(\lambda). \tag{1.13}$$

Therefore we can give the following Definition.

Definition 1.2.9 (Direct integral Hilbert space)**.** *Suppose the following structures are given:*

1. *a measure space (X, Ω, μ) with μ a σ-finite measure;*
2. *a collection $\{\mathcal{H}_\lambda\}_{\lambda\in X}$ of separable Hilbert spaces for which the dimension function $\Psi : \lambda \mapsto \dim(\mathcal{H}_\lambda)$ is measurable;*
3. *a measurability structure on $\{\mathcal{H}_\lambda\}_{\lambda\in X}$, i.e. a choice of simultaneous orthonormal basis.*

Then, the direct integral of the $\{\mathcal{H}_\lambda\}_{\lambda\in X}$ with respect to μ, denoted

$$\int_X^{\oplus} \mathcal{H}_\lambda d\mu(\lambda),$$

is the space of equivalence classes of almost-everywhere equal measurable sections s for which

$$||s||^2 := \int_X \langle s(\lambda), s(\lambda)\rangle_\lambda d\mu(\lambda) < +\infty.$$

This notion is a generalization of the direct sum of Hilbert spaces from finite, or discrete, sets of labels to labels which vary continuously and parametrized by a measure space (X, Ω, μ). We are now ready to state the spectral theorem in its second form.

Theorem 1.2.2 (Spectral theorem II)**.** *Suppose O is a self-adjoint operator on $\mathcal{H}$. Then there is a σ-finite measure μ on $\sigma(O)$, a direct integral Hilbert space*

$$\int_{\sigma(O)}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)$$

and a unitary map U from $\mathcal{H}$ to the direct integral Hilbert space such that

$$U(\mathrm{Dom}(O)) = \left\{ s \in \int_{\sigma(O)}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda) \;\middle|\; \int_{\sigma(O)} ||\lambda s(\lambda)||^2 d\mu(\lambda) < +\infty \right\},$$

and such that

$$(UOU^{-1}(s))(\lambda) = \lambda s(\lambda) \qquad \forall\ s \in U(\mathrm{Dom}(O)).$$

Moreover, we can recast the spectral theorem in a multiplication operator form.

Theorem 1.2.3 (Spectral theorem II+)**.** *Suppose O is a self-adjoint operator on $\mathcal{H}$. Then there is a measure space (X, Ω, μ) with μ a σ-finite measure, a measurable real-valued function h on X and a unitary map $\tilde{U} : \mathcal{H} \to L^2(X, \mu)$ such that*

$$\tilde{U}(\mathrm{Dom}(O)) = \left\{ \psi \in L^2(X, \mu) \mid h\psi \in L^2(X, \mu) \right\},$$

and such that

$$(\tilde{U}O\tilde{U}^{-1}(\psi))(x) = h(x)\psi(x) \qquad \forall\ \psi \in \tilde{U}(\mathrm{Dom}(O)).$$

The conclusion is that up to pre-compose and post-compose with a unitary operator $\tilde{U}$, the operator O is the multiplication by the h function in a L^2-space that is unitarily equivalent to the original Hilbert space $\mathcal{H}$. Same considerations hold for the case of the second form of the spectral theorem, Theorem 1.2.2. In that case, $\mathcal{H}$ is unitarily equivalent to a direct integral Hilbert space and the action of the operator O is, up to pre-compose and post-compose with a unitary operator U, the multiplication by the function $h(\lambda) = \lambda$.

1.2.4 Stone's theorem and one-parameter unitary groups

Here we explore the notion of one-parameter unitary groups and their connection to self-adjoint operators. We start with the following Definition

Definition 1.2.10 (One-parameter unitary group)**.** *A one-parameter unitary group on $\mathcal{H}$ is a family $U(t)$ with $t \in \mathbb{R}$ of unitary operators with the property that $U(0) = 1$ and that $U(s+t) = U(s)U(t)$ for all $s, t \in \mathbb{R}$. A one-parameter unitary group is said to be strongly continuous if*

$$\lim_{s \to t} ||U(t)\psi - U(s)\psi|| = 0 \qquad \forall \psi \in \mathcal{H}, t \in \mathbb{R}.$$

The basic example of a one-parameter unitary group is the one of translations in $L^2(\mathbb{R}^n)$. Indeed, let $U_a(t)$ with $a \in \mathbb{R}^n$ be the translation operator given by $(U_a(t)\psi)(x) = \psi(x+ta)$ then the family $U(t)$ is a strongly continuous one-parameter unitary group.
The goal of this Paragraph is to state Stone's theorem that is the quantum analog of Nöther's theorem [11]. Thanks to functional calculus we can define the operator

$$U(t) := e^{itO} \qquad t \in \mathbb{R}, \tag{1.14}$$

where O is a self-adjoint unbounded operator on $\mathcal{H}$. The following proposition asserts that the operator (1.14) gives rise to a one-parameter unitary group.

Proposition 1.2.3 (Self-adjoint operators and one-parameter unitary group)**.** *Suppose O is a self-adjoint operator on $\mathcal{H}$ and let $U(t)$ be defined by* (1.14)*. Then $U(t)$ with $t \in \mathbb{R}$ is a strongly continuous one-parameter unitary group and O, called its infinitesimal generator, is given by*

$$O\psi = \lim_{t\to 0} \frac{1}{i}\frac{U(t)\psi - \psi}{t} \qquad \forall\ \psi \in \mathrm{Dom}(O)$$

in the norm topology of $\mathcal{H}$.

Proof. Since O is self-adjoint, its spectrum is contained in the real line by Proposition 1.2.1. Hence the function $f(\lambda) := e^{it\lambda}$ with $\lambda \in \sigma(O) \subset \mathbb{R}$ is bounded on $\sigma(O)$ and satisfies $f(\lambda)f(\lambda)^* = 1$. Thus, the operator $f(O)$ is bounded and satisfies $f(O)f(O)^* = f(O)^*f(O) = 1$ which shows that $U(t) := f(O) = e^{itO}$ is unitary. The multiplicativity of the functional calculus then shows that this is a one-parameter unitary group. To see that $U(t)$ is strongly continuous, note that

$$||U(t)\psi - U(s)\psi|| = \langle \psi (U^*(t) - U^*(s))(U(t) - U(s))\psi\rangle = \int_{-\infty}^{+\infty} |e^{it\lambda} - e^{is\lambda}|^2 d\mu_\psi^O(\lambda),$$

that tends to zero as s approaches t, by dominated convergence. In a similar manner we can show the form of the infinitesimal generator. □

We can now state the Stone's theorem whose proof can be found in [1].

Theorem 1.2.4 (Stone)**.** *Suppose $U(t)$ with $t \in \mathbb{R}$ is a strongly continuous one-parameter unitary group on $\mathcal{H}$. Then its infinitesimal generator O is densely defined and self-adjoint, and $U(t) = e^{itO}$ for all $t \in \mathbb{R}$.*

At this point we can briefly discuss the solution of the Schrödinger equation (1.5); from the theory discussed above the solution is given by

$$\psi(t) = G(t)\psi_0 := e^{-itH}\psi_0 \tag{1.15}$$

where $G(t)$ defines a strongly continuous one-parameter group of unitary operators on $\mathcal{H}$, called the unitary propagator. Moreover, $G(t)$ is linear and isometric for all $t \in \mathbb{R}$. In a similar way, the solution to the Heisenberg equation (1.6) is given by

$$O(t) = G^{-1}(t)O_0G(t). \tag{1.16}$$

The two pictures are related by the fundamental Ehrenfest's theorem.

Theorem 1.2.5 (Ehrenfest)**.** *Let $O = O^*$ be a quantum mechanical observable and $[\psi] \in \mathbb{P}\mathcal{H}$, then*

$$\langle O \rangle_{[\psi(t)]} = \langle O(t) \rangle_{[\psi]}.$$

Proof. It is enough to compute

$$\langle \psi(t) | O\psi(t) \rangle = \langle G(t)\psi | OG(t)\psi \rangle = \langle \psi | G^{-1}(t) OG(t)\psi \rangle = \langle \psi | O(t)\psi \rangle.$$

since $G^*(t) = G^{-1}(t)$. □

Therefore the two pictures are unitarily equivalent. However, we can build up also unitarily inequivalent pictures of Quantum Mechanics that are interesting for cosmological applications [12]; in Appendix A we briefly discuss the basic starting point for the definition of one of such unitarily inequivalent pictures called Polymer Quantum Mechanics [9, 10].

1.3 Symmetries in Quantum Mechanics

Certain transformations leave invariant the physical system: these are then called symmetries of the system. In the classification of topological phases of matter, the symmetries present in the quantum system are of fundamental importance.

1.3.1 Quantum symmetries generalities

We have already stressed the probabilistic interpretation of Quantum Mechanics: the state of the quantum particle is linked to the probability density of its position, while the evolution of a state is dictated deterministically by the Schrödinger equation. It is a matter of postulates that after the measurement of the observable properties of the state $[\psi]$, the system "collapses" into a new state which is determined only probabilistically. In particular, the transition probability from the state $[\psi]$ to a state $[\phi]$ is postulated to be

$$\mathrm{Prob}([\psi] \mapsto [\phi]) := |\langle \phi | \psi \rangle|^2. \tag{1.17}$$

From the physical point of view this is the quantity we want to be preserved by symmetries. However, we can also give a geometric, in fact projective, interpretation to the transition amplitude (1.17); indeed defining the pseudo-angle

$$\mathbb{P}([\psi], [\phi]) := \frac{|\langle \psi | \phi \rangle|}{||\psi|| ||\phi||} \tag{1.18}$$

in the projective Hilbert space $\mathbb{P}\mathcal{H}$, a symmetry transformation $T : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ has to preserve it

$$\mathbb{P}(T([\psi]), T([\phi])) = \mathbb{P}([\psi], [\phi]). \tag{1.19}$$

Hence, we start with the following definition of symmetry.

Definition 1.3.1 (Wigner symmetry)**.** *A symmetry is a surjective transformation $T : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ such that*

$$|\langle \phi | \psi \rangle| = |\langle \phi' | \psi' \rangle| \qquad \text{if} \quad [\phi'] = T([\phi]),\ [\psi'] = T([\psi]).$$

In view of a powerful theorem we are going to show, we need the definition of unitary and anti-unitary transformations.

Definition 1.3.2 (Unitary and anti-unitary transformations)**.** *A transformation $U : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ is called unitary (anti-unitary) if*

1. $U(\psi + \phi) = U(\psi) + U(\phi) \qquad \forall\ \psi, \phi \in \mathcal{H}$;
2. $U(z\psi) = \tilde{z}U(\psi) \qquad \forall\ z \in \mathbb{C}$ and $\psi \in \mathcal{H}$.
   We have $\tilde{z} = z$ for U unitary and $\tilde{z} = z^*$ for U anti-unitary;
3. $\langle U\psi|U\phi\rangle = \widetilde{\langle\psi|\phi\rangle} \qquad \forall\ \psi, \phi \in \mathcal{H}$.
   We have $\widetilde{\langle\psi|\phi\rangle} = \langle\psi|\phi\rangle$ for U unitary and $\widetilde{\langle\psi|\phi\rangle} = \langle\phi|\psi\rangle$ for U anti-unitary;
4. *U is surjective.*

The fundamental result about the Wigner symmetry is the following [13, 14, 15].

Theorem 1.3.1 (Wigner)**.** *Any symmetry T is of the form $T \equiv T_U$ for U unitary or anti-unitary.*

A complete proof is given in Appendix B. A useful corollary of the above theorem is the following.

Corollary 1.3.1 (U(1)-equivalence of the lifted symmetry)**.** *Let $\mathcal{H}$ be an Hilbert space and $\dim(\mathcal{H}) \geq 2$. Let T be a symmetry, and assume that it is lifted by both $U_1 : \mathcal{H} \to \mathcal{H}$ and $U_2 : \mathcal{H} \to \mathcal{H}$. Then $U_2 = e^{i\theta}U_1$ for some $e^{i\theta} \in \mathrm{U}(1)$. In particular, the transformations of $\mathcal{H}$ lifting a symmetry are either all unitary or all anti-unitary.*

At the level of observables, symmetries are essentially induced by how the symmetries act at the level of states by requiring that for all observable $\langle O\rangle_{T([\psi])} = \langle\tilde{T}(O)\rangle_{[\psi]}$. By Wigner theorem we have that $T \equiv T_U$ with U unitary or anti-unitary, hence

$$\langle\tilde{T}(O)\rangle_{[\psi]} = \langle O\rangle_{T([\psi])} = \langle U(\psi)|OU(\psi)\rangle = \langle U(\psi)|UU^{-1}OU(\psi)\rangle = \langle\psi|U^{-1}OU(\psi)\rangle \tag{1.20}$$

where the last equality holds since $U^{-1}OU$ is self-adjoint if O is so, hence

$$\langle\psi|U^{-1}OU(\psi)\rangle \in \mathbb{R}$$

and it is irrelevant whether U is unitary or anti-unitary. By definition we have $\langle\psi|\tilde{T}(O)(\psi)\rangle = \langle\tilde{T}(O)\rangle_{[\psi]}$, therefore we must have

$$\tilde{T}(O) = U^{-1}OU. \tag{1.21}$$

The particular case in which O is the Hamiltonian H of the system takes the name of dynamical symmetry. Hence we have the following Definition.

Definition 1.3.3 (Dynamical symmetry)**.** *A symmetry $\tilde{T}$ is called a dynamical symmetry if the associated operator U commutes with the Hamiltonian H, i.e. $[U, H] = 0$.*

1.3.2 Algebraic structure of quantum symmetries

Symmetries in physics are such that different symmetry transformations can be composed to yield new symmetries; hence they have the algebraic structure of a group $(\mathrm{G}, \cdot)$. In many cases of physical interest, the set of symmetries has not only the group structure but also a differentiable structure which makes it a Lie group.

Definition 1.3.4 (Lie group and topological group)**.** *A Lie group* $(\mathrm{G}, \cdot)$ *is a differentiable manifold which is endowed with a group structure such that the group operations*

1 $\cdot : \mathrm{G} \times \mathrm{G} \to \mathrm{G}, \quad (g_1, g_2) \mapsto g_1 \cdot g_2$

2 ${}^{-1} : \mathrm{G} \to \mathrm{G}, \quad g \mapsto g^{-1}$

are differentiable. If G *is only a topological manifold and the group operations are only continuous, then* $(\mathrm{G}, \cdot)$ *is called a topological group. The dimension of a topological group or a Lie group* $(\mathrm{G}, \cdot)$ *is defined to be the dimension of* G *as a manifold.*

Let us consider the example of $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. Given three elements $x, y, z \in \mathbb{R}^*$ such that $xy = z$, if we multiply a number close to x by a number close to y, we obtain a number close to z. Similarly, the inverse of a number close to x is close to $1/x$. In fact, we can differentiate these maps and $\mathbb{R}^*$ is made into a Lie group with these group operations. Other examples are provided by matrix groups, which are subgroups of general linear groups $\mathrm{GL}(m, \mathbb{R})$ or $\mathrm{GL}(m, \mathbb{C})$, where the product of elements is simply the matrix multiplication and the inverse is given by the matrix inverse. These are the special linear groups $\mathrm{SL}(m, \mathbb{R})$ and $\mathrm{SL}(m, \mathbb{C})$, the orthogonal group $\mathrm{O}(m)$ and the special orthogonal group $\mathrm{SO}(m)$, the unitary group $\mathrm{U}(m)$ and the special unitary group $\mathrm{SU}(m)$. In general we have the following useful criterion whose proof can be found in [16].

Theorem 1.3.2 (Cartan criterion)**.** *Every closed subgroup* H *of a Lie group* G *is a Lie subgroup.*

Since a Lie group is also a differentiable manifold we can consider its tangent space to a point g. On any tangent space of a Lie group $(\mathrm{G}, \cdot)$ are defined two natural maps induced by the right-translation and the left-translation

$$\begin{aligned} R_h &: \mathrm{G} \to \mathrm{G}, \quad g \mapsto g \cdot h; \\ L_h &: \mathrm{G} \to \mathrm{G}, \quad g \mapsto h \cdot g. \end{aligned} \tag{1.22}$$

By definition, R_h and L_h are diffeomorphisms from G to G, hence they induce, by pushforward,

$$(L_a)_* : T_g\mathrm{G} \to T_{h \cdot g}\mathrm{G} \quad \text{and} \quad (R_a)_* : T_g\mathrm{G} \to T_{g \cdot h}\mathrm{G}.$$

On a generic differentiable manifold there is no canonical way of discriminating some vector fields from the others but, thanks to the group structure, if we have a Lie group $(\mathrm{G}, \cdot)$, there exists a special class of vector fields characterized by an invariance under group action.

Definition 1.3.5 (Left-invariant vector fields)**.** *Let W be a vector field on a Lie group* $(\mathrm{G}, \cdot)$*. W is said to be a left-invariant vector field if*

$$(L_h)_*(W_g) = W_{h\cdot g}.$$

A vector $W_e \in T_e\mathrm{G}$ defines a unique left-invariant vector field W throughout G by

$$W_g := (L_g)_*(W_e), \qquad g \in \mathrm{G} \tag{1.23}$$

In fact

$$W_{h\cdot g} = (L_{h\cdot g})_*(W_e) = (L_h L_g)_*(W_e) = (L_h)_*(L_g)_*(W_e) = (L_h)_*(W_g). \tag{1.24}$$

Conversely, a left-invariant vector field W defines a unique vector $W_e \in T_e\mathrm{G}$. The set of left-invariant vector fields on $(\mathrm{G}, \cdot)$ is denoted $\mathfrak{g}$. An important result of the Lie group theory states the map

$$\Phi : T_e\mathrm{G} \to \mathfrak{g}, \quad W_e \mapsto W \tag{1.25}$$

is an isomorphism. It follows that the set of left-invariant vector fields $\mathfrak{g}$ is a vector space isomorphic to $T_e\mathrm{G}$, so $\dim(\mathfrak{g}) = \dim(\mathrm{G})$. Since $\mathfrak{g}$ is a set of vector fields it can be endowed by Lie bracket and it is closed under this operation as the following Proposition shows.

Proposition 1.3.1 (Closure of $\mathfrak{g}$ under Lie bracket)**.** *The vector space $\mathfrak{g}$ is closed under Lie bracket operation; hence it is an anti-commutative algebra.*

Proof. Take two points g and $h \cdot g = L_h(g)$ in G. If we apply $(L_h)_*$ to the Lie bracket $[W, Z]$ of $W, Z \in \mathfrak{g}$, we have

$$(L_h)_*([W, Z]_g) = [(L_h)_*(W_g), (L_h)_*(Z_g)] = [X, Y]_{h\cdot g}$$

where the left-invariances of X and Y, and the pushforward property with respect to the Lie bracket have been used. Thus, $[W, Z] \in \mathfrak{g}$. □

We can now define the Lie algebra of the Lie group $(\mathrm{G}, \cdot)$

Definition 1.3.6 (Lie algebra)**.** *The vector space $\mathfrak{g}$ with the Lie bracket*

$$[\cdot, \cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is called the Lie algebra of a Lie group $(\mathrm{G}, \cdot)$*. The generators of the Lie algebra are the base elements $\mathfrak{X}_1, ..., \mathfrak{X}_{\dim(\mathfrak{g})}$ of the underlying vector space and the structure constants of a Lie algebra are the coefficients $\{f_{ij}^k\}$ that appear in the writing of the Lie bracket between the elements of a base of the algebra with respect to the same base*

$$[\mathfrak{X}_i, \mathfrak{X}_j] = \sum_{k=1}^{\dim(\mathfrak{g})} f_{ij}^k \, \mathfrak{X}_k.$$

Some Lie groups and Lie algebras properties can be efficiently studied by using representations.

Definition 1.3.7 (Group and algebra representation)**.** *Let* $(\mathrm{G}, \cdot)$ *be a group. A representation of* $(\mathrm{G}, \cdot)$ *on a vector space* V *is a group homomorphism*

$$\rho : \mathrm{G} \to \mathrm{GL}(V).$$

If $(\mathrm{G}, \cdot)$ *is a Lie group and* $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ *its Lie algebra, then a representation of* $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ *on a vector space* V *is a Lie algebra homomorphism*

$$\tilde{\rho} : \mathfrak{g} \to \mathfrak{gl}(V).$$

An important example of Lie group representation is the so-called ajoint representation.

Definition 1.3.8 (Adjoint representations)**.** *Let* $(\mathrm{G}, \cdot)$ *be a Lie group and* $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ *its Lie algebra. The adjoint representation of* $(\mathrm{G}, \cdot)$ *is the Lie group homomorphism*

$$\mathrm{Ad} : \mathrm{G} \to \mathrm{GL}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g$$

such that

$$\mathrm{Ad}_g(\mathfrak{X}) = g\mathfrak{X}g^{-1}, \qquad \forall\ g \in \mathrm{G}, \mathfrak{X} \in \mathfrak{g}.$$

The adjoint representation of $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ *is the Lie algebra homomorphism given by the differential of the adjoint representation of* $(\mathrm{G}, \cdot)$ *at the identity*

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad \mathfrak{X} \mapsto \mathrm{ad}_{\mathfrak{X}} = d(\mathrm{Ad})_e(\mathfrak{X})$$

such that

$$\mathrm{ad}_{\mathfrak{X}}(\mathfrak{Y}) = [\mathfrak{X}, \mathfrak{Y}], \qquad \forall\ \mathfrak{X}, \mathfrak{Y} \in \mathfrak{g}.$$

Therefore the adjoint representation of $(\mathrm{G}, \cdot)$ is a representation on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ of the group while the adjoint representation of $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ is a representation on the Lie algebra itself.

Lie groups and Lie algebras are fundamental in theoretical physics as they provide a rigorous mathematical framework for describing differentiable symmetries. These symmetries underlie many physical principles, from conservation laws to the structure of fundamental interactions.
In classical and quantum mechanics, physical systems often exhibit invariance under groups such as $\mathrm{SO}(3)$ (orbital rotations), $\mathrm{SU}(2)$ (spin), and $\mathrm{SO}(1,3)$ (Lorentz transformations in 4D). The Lie algebra encodes commutation relations among generators of these symmetries; for instance, the orbital angular momentum operators in 3D, $\{J_x, J_y, J_z\}$, satisfy $[J_i, J_j] = \sqrt{-1}\epsilon_{ijk}J_k$, forming a representation of $\mathfrak{so}(3)$.
In Quantum Field Theory, local gauge invariance under a Lie group gives rise to gauge theories, with the Standard Model being the most prominent example. It is based on the gauge group whose manifold is $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, describing strong, weak, and electromagnetic interactions. The corresponding gauge bosons (gluons, $W^{\pm}$, Z, and the photon) transform under the adjoint representation of their respective Lie algebras. The structure constants of the Lie algebra enter directly into the gauge field strength and hence into the Feynman rules, affecting the interaction vertices in perturbative calculations and determining selection rules.
Lie groups and their representations are also pivotal in String Theory: to be

consistent, superstring theories require anomaly cancellation, which restricts the gauge group to specific Lie groups such as SO(32) or $E_8 \times E_8$ and forces particular string interactions.
In Condensed Matter Physics, Lie groups and Lie algebras describe both internal and spatial symmetries of many-body systems. The effective low-energy dynamics of systems with spontaneous symmetry breaking, such as ferromagnets or superfluids, is described by nonlinear sigma models with target spaces of the form G/H, where G is the symmetry group and H the unbroken subgroup. The Goldstone bosons correspond to fluctuations along these coset directions.
These Lie algebraic structures also appear in more advanced frameworks such as the BRST formalism, where ghost fields carry representations of the gauge algebra, and in the study of anomalies, where the cohomology of the Lie algebra plays a crucial role.
However, not all physical information is encoded in the Lie algebra alone. Many important non-perturbative effects depend on the global topology of the Lie group or the gauge bundle. For instance, instantons, which are finite action solutions in non-abelian gauge theories, are classified by the higher homotopy group $\pi_3(\mathrm{G})$. These effects lead to tunneling between vacua and contribute to phenomena like the θ-vacuum structure in Quantum ChromoDynamics and the resolution of the U(1) axial anomaly. Furthermore, topological excitations such as vortices and skyrmions are classified using homotopy groups of G/H, again requiring knowledge of the global structure of the group.
Finally, Lie groups also play an essential role in cosmology, classical gravity theories in general and in biophysics. On the one hand, an interesting theorem about pseudo-Riemannian manifolds states that the isometry group of a pseudo-Riemannian manifold is a Lie group. Hence all the symmetries of space-time manifolds form a Lie group and so, locally, are described by Lie algebras. On the other hand, Lie groups and Lie algebras are used to model and analyze complex biological systems, such as protein dynamics, cell signaling and neural networks.

Back to quantum systems, since the space of physical states is a projective one we are interested in projective representations which give us information about the structure of the symmetry group G through the action of symmetries on the space of physical states $\mathbb{P}\mathcal{H}$.

Definition 1.3.9 (Projective representation)**.** *Let* $(\mathrm{G}, \cdot)$ *be an abstract group. A projective representation of* $(\mathrm{G}, \cdot)$ *on the Hilbert space* $\mathcal{H}$ *is a collection of maps*

$$\{\rho_g : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}\}_{g \in \mathrm{G}}$$

such that

1. ρ_g *is a symmetry for every* $g \in \mathrm{G}$*;*
2. $\rho_e = 1_{\mathbb{P}\mathcal{H}}$*;*
3. *the map* $\Phi : g \mapsto \rho_g$ *is a homomorphism.*

Since $\{\rho_g\}_{g \in \mathrm{G}}$ are symmetries, by Wigner theorem we can associate to them an unitary or anti-unitary operator $U_g : \mathcal{H} \to \mathcal{H}$. From the homomorphism property of a projective representation it follows that if $\psi \in \mathbb{S}(\mathcal{H})$ and $g, h \in \mathrm{G}$ we have

$$[U_{g\cdot h}(\psi)] = \rho_{g\cdot h}([\psi]) = \rho_g \rho_h([\psi]) = [U_g U_h(\psi)] \quad \Rightarrow \quad U_{g\cdot h} = \omega(g,h) U_g U_h; \quad (1.26)$$

where $\omega(g,h) : \mathrm{G} \times \mathrm{G} \to \mathrm{U}(1)$. Moreover the associativity condition forces the identity

$$\omega(g\cdot h, k)\omega(g,h) = \omega(g, h\cdot k)\omega(h,k) \qquad \forall\ g,h,k \in \mathrm{G}. \quad (1.27)$$

We can recognize $\omega(g,h)$ as a U(1)-valued 2-cocycle on G. To see this let us briefly introduce group cohomology [17, 18]. This is a tool similar to group representations, that looks at the group actions of a group G in an associated G-module or (G,G)-bimodule M to elucidate the properties of the group. By treating the G-module as a kind of topological space with elements of $\mathrm{G}^{\times n}$ representing n-simplices, topological properties of the space may be computed. For $n \geq 0$ let $C^n(\mathrm{G}, M)$ be the group, in additive notation, of all functions from $\mathrm{G}^{\times n}$ to M. In the following we are interested in maps from $\mathrm{G}^{\times n}$ to U(1) so we consider the G-module M as the abelian group U(1) with scalar multiplication $\cdot : \mathrm{G} \times \mathrm{U}(1) \to \mathrm{U}(1)$ given by the identity[3]. In this context, the coboundary homomorphisms $d^{n+1} : C^n(\mathrm{G}, \mathrm{U}(1)) \to C^{n+1}(\mathrm{G}, \mathrm{U}(1))$ are defined by

$$\left(d^{n+1}\varphi\right)(g_1, ..., g_{n+1}) := \sum_{i=0}^{n+1} (-1)^i \varphi\left(g_1, \ldots, g_{i-1}, g_i \cdot g_{i+1}, g_{i+2}, \ldots, g_{n+1}\right) \quad (1.28)$$

We may check that $d^{n+1} \circ d^n = 0$, so this defines a cochain complex whose cohomology can be computed in the usual way. If $\varphi(g_1, g_2) = \omega(g,h)$ is a 2-cocycle then from (1.28) we have

$$(d^3\omega)(g,h,k) = \omega(h,k) - \omega(g\cdot h, k) + \omega(g, h\cdot k) - \omega(g,h) = 0, \quad (1.29)$$

that, in multiplicative notation, is

$$\omega(h,k)\omega(g,h\cdot k)\omega^{-1}(g\cdot h,k)\omega^{-1}(g,h) = 1, \quad (1.30)$$

from which condition (1.27) derives.

We are interested in better understanding the lift from the projective representations to the unitary (or anti-unitary) maps U_g. In this perspective, we first give the following Definition.

Definition 1.3.10 (Projective unitary representation)**.** *A projective unitary representation of the group* $(\mathrm{G}, \cdot)$ *on the Hilbert space* $\mathcal{H}$ *is a map* $U : \mathrm{G} \to \mathcal{U}(\mathcal{H})$ *such that the multiplier operators are multiples of the identity*

$$U_{g\cdot h}^{-1} U_g U_h = \omega(g,h) 1_{\mathcal{H}}$$

where $\omega(g,h)$ *is a* U(1)*-valued 2-cocycle on* G*. If* $\omega(g,h) \equiv 1$ *then* U *is called a unitary representation.*

[3] If M is a (G,G)-bimodule we consider it as the abelian group U(1) with left scalar multiplication $\cdot : \mathrm{G} \times \mathrm{U}(1) \to \mathrm{U}(1)$ and right scalar multiplication $\cdot\cdot : \mathrm{U}(1) \times \mathrm{G} \to \mathrm{U}(1)$ both given by the identity.

A similar definition can be done in the anti-unitary case. However, we stress that since the composition of two anti-unitary maps is unitary, if G has more than two elements it cannot be represented projectively on a Hilbert space solely by anti-unitary operators plus the identity. In particular, if $(\mathrm{G}, \cdot)$ is a connected Lie group, then all operators that lift a projective representation must be unitary. Therefore, anti-unitary operators occur as lifts in projective representations only if G is discrete and contains $\mathbb{Z}_2$ as a factor, or if G is not connected and has $\mathbb{Z}_2$ as a quotient by a connected normal subgroup.
The crucial point is that, under topological conditions, it is possible to choose $\omega(g,h) \equiv 1$. Before, we need the following Definition.

Definition 1.3.11 (Equivalence projective unitary representations)**.** *Two projective unitary representations $U, V : \mathrm{G} \to \mathcal{U}(\mathcal{H})$ are said to be equivalent if there exist $S \in \mathcal{U}(\mathcal{H})$ and $\chi : \mathrm{G} \to \mathrm{U}(1)$ such that*

$$SU_gS^{-1} = \chi(g)V_g \qquad \forall\ g \in \mathrm{G}.$$

We have the following result.

Theorem 1.3.3. *A projective unitary representation U of the group $(\mathrm{G}, \cdot)$ is equivalent to a unitary one V if and only if the $\mathrm{U}(1)$-valued 2-cocycle ω is a $\mathrm{U}(1)$-valued 2-coboundary. Hence, equivalence classes of projective unitary representations of the group $(\mathrm{G}, \cdot)$ are thus in one to one correspondence with the second group cohomology group $H^2(\mathrm{G}; \mathrm{U}(1))$.*

Proof. Let us first assume that U and V are equivalent. Hence, there exist $S \in \mathcal{U}(\mathcal{H})$ and $\chi : \mathrm{G} \to \mathrm{U}(1)$ such that for all $g, h \in \mathrm{G}$

$$\begin{aligned} U_{g\cdot h}^{-1}U_gU_h &= (\chi(g\cdot h)S^{-1}V_{g\cdot h}S)^{-1}(\chi(g)S^{-1}V_gS)(\chi(h)S^{-1}V_hS) = \\ &= \chi(g\cdot h)^{-1}\chi(g)\chi(h)S^{-1}(V_{g\cdot h}^{-1}V_gV_h)S = \chi(g\cdot h)^{-1}\chi(g)\chi(h)1_{\mathcal{H}} \end{aligned}$$

since $V_{g\cdot h}^{-1}V_gV_h = 1_{\mathcal{H}}$ because V is a unitary representation. However, by definition $U_{g\cdot h}^{-1}U_gU_h = \omega(g,h)1_{\mathcal{H}}$, so $\omega(g,h) = \chi(g\cdot h)^{-1}\chi(g)\chi(h)$ that, from (1.28), is the coboundary condition.
On the other way around, if $\omega(g,h)$ is a U(1)-valued 2-coboundary we define $V_g = \chi^{-1}(g)U_g$ for $g \in \mathrm{G}$, then

$$V_{g\cdot h}^{-1}V_gV_h = \chi(g\cdot h)\chi^{-1}(g)\chi^{-1}(h)U_{g\cdot h}U_gU_h = \omega(g,h)^{-1}\omega(g,h) = 1_{\mathcal{H}}.$$

□

Chapter 2

Bundle theory and condensed matter

Relevant symmetries of quantum systems often arise as spatial symmetries of the corresponding configuration space. This is the case, for example, of translations or rotations. Translational symmetry is described by saying that the configuration space X of the quantum particle is acted upon by an abelian group G of translations. In Condensed Matter Physics, there are two prominent examples: a continuum quantum system whose translation group is $(\mathbb{R}^d, +)$ or a subgroup of it and a discrete quantum system whose translation group is a lattice. In these cases, we have the possibility of finding a privileged basis of eigenfunctions, possibly generalized, which "diagonalizes" or "partially diagonalizes" Hamiltonian and translation operators acting on the Hilbert space of the model. This goal is achieved by the Fourier transform (in the case where the translation group is $(\mathbb{R}^d, +)$) or by the Zak-Bloch-Floquet transform (in the cases where the translation group is a lattice of $(\mathbb{R}^d, +)$). In the second case, which will be the one of interest to us, the Hamiltonian is decomposed into fibered operators on a direct integral Hilbert space whose underlying measure space is that of the quasi-momentum, i.e. the Brillouin torus. The generalized eigenfunctions of these fibered operators are the so-called Bloch eigenfunctions and the spectrum of the original Hamiltonian can be reconstructed by looking at the spectra of these fibered operators: this gives the so-called band-gap theory of condensed matter and solid state physics. Under the hypothesis of a spectral gap, i.e. of non-overlapping bands, we can describe an insulating material. The eigenspaces associated to the set of m bands can be put in correspondence with the associated spectral eigenprojectors thanks to which it is possible to organize this information in a vector bundle called Bloch bundle. In this way we are able to recover topological information from a condensed matter system.

In Section 2.1 we introduce condensed matter system by defining the Wigner-Seitz cell and the Brillouin torus, that, endowed by the Lebesgue measure, will be the measure space we are interested in since it is the space where the quasi-momentum lives. We also introduce the concepts of Bloch functions, Bloch Hamiltonians and Bloch bands.

In Section 2.2, with the goal to "partially diagonalize" Hamiltonian operators invariant under a lattice of translations, we define the Zak-Bloch-Floquet transform.

We prove an extension theorem for the Zak-Bloch-Floquet operator and we discuss how the Hamiltonian decomposes in fibered operator. This opens the doors to the band-gap theory where the spectrum of the Hamiltonian operator is given by an alternation of spectral bands and gaps; the total spectrum is reconstructed as the union of all bands. In the end, we could draw the spectrum of the Hamiltonian as a numerable set of function graphs. The topological properties are encoded in the valence energy bands, hence we introduce spectral eigenprojections that can be expressed by Riesz formula. The ranges of these spectral eigenprojections can be organized in a suitable way to give rise to the Bloch bundle which is a vector bundle that encodes the topological content of the condensed matter system.

In Section 2.3, motivated by the emergence of the Bloch bundle, we focus on bundle theory introducing principal bundles and associated bundles; we also study connections, parallel transports and curvature on principal and associated vector bundles. Furthermore, in view of the homotopy classification of vector bundles we introduce Chern classes and Chern-Weil theory which allows to present the Chern classes as differential forms starting from the curvature of the bundle.

In Appendix E, we perform the homotopy classification of complex vector bundles and we discuss the cohomology ring of the infinite Grassmanian.

The main references for this Chapter are [5, 6, 19, 20, 21, 22, 23, 24].

2.1 Introduction to condensed matter systems

In solids the ionic cores, whose positions can be assumed to be fixed in view of the Born-Oppenheimer approximation [25], form a crystal, that is, a regular arrangement in space which repeats in all directions. Electrons move in this background. To model, in the continuum limit, the quantum dynamics of an electron in a d-dimensional crystal, we need a Hamiltonian

$$H = -\frac{1}{2m}\Delta + V_\Gamma \tag{2.1}$$

which is periodic with respect to a subgroup of the entire translation group $(\mathbb{R}^d, +)$. This subgroup is called Bravais lattice of translations

$$\Gamma := \mathrm{Span}_{\mathbb{Z}}(a_1, ..., a_d) \simeq \mathbb{Z}^d \subset \mathbb{R}^d, \tag{2.2}$$

where $a_1, ...a_d$ is a basis of $\mathbb{R}^d$ and it is in general different from the regular arrangement of positions of the ionic cores. The periodicity condition means that translations by vectors $\gamma \in \Gamma$ are implemented as operators U_γ acting unitarily on the Hilbert space $\mathcal{H}$ of the electron such that $\tilde{T}_\gamma$, that are the symmetry transformation induced at the level of observables by T_{U_γ}, are dynamical symmetries, i.e.

$$[U_\gamma, H] = 0. \tag{2.3}$$

Moreover, the map $U : \Gamma \to \mathcal{U}(\mathcal{H})$ that maps $\gamma \mapsto U_\gamma$ must be a unitary representation of Γ on $\mathcal{H}$. As usual, translations are implemented as

$$(U_\gamma(\psi))(x) = \psi(x + \gamma), \qquad \forall\ \psi \in L^2(\mathbb{R}^d),\ \gamma \in \Gamma; \tag{2.4}$$

which forces $V_\Gamma(x+\gamma) = V_\Gamma(x)$ for all $\gamma \in \Gamma$. It is often the case that Condensed Matter Physics deals with a discrete model of a crystal, in which each cell of the Bravais lattice contains only a finite number $N \in \mathbb{N}$ of degrees of freedom. Electrons can then only jump among lattice sites, as if they were bound to their positions: this is the so-called tight-binding approximation [26, 27]. Therefore, the Hilbert space of the electron is then

$$\mathcal{H} = \ell^2(\Gamma) \otimes \mathbb{C}^N \simeq \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N, \tag{2.5}$$

and the action of the Hamiltonian is given by

$$(H(\psi))_x = \sum_{y\in\Gamma} t_{x,y}\psi_y + V_x\psi_x, \qquad \psi \in \ell^2(\Gamma),\ x \in \Gamma. \tag{2.6}$$

The coefficient $t_{x,y}$ is interpreted as the hopping amplitude for the electron to jump from site x to site y, while V_x represents an on-site potential energy. Lattice translations are implemented as

$$(U_\gamma(\psi))_x = \psi_{x+\gamma}, \qquad \psi \in \ell^2(\Gamma),\ x, \gamma \in \Gamma. \tag{2.7}$$

In the following we are going to present the material in the continuum case but what we will say translates immediately to the case of the tight-binding approximation, i.e. the discrete case. We return to tight binding models in Chapter 4.

A general paradigm in Physics is that if a system has some symmetry, we can use it to simplify the analysis of its properties. Therefore, we are going to use lattice translation symmetry of periodic Hamiltonians to investigate their spectral properties gaining a "partial diagonalization" (in the sense of the spectral theorem) of such operators, i.e. a set of diagonalized operators labeled by the so-called quasi-momentum. First of all, let us note that the operators U_γ commute since Γ is an abelian group. Hence, we can simultaneously diagonalize the operators U_γ and we can choose a common generalized eigenvector ψ for all the translation operators such that

$$(U_\gamma(\psi))(x) = \chi_\gamma\psi(x) \qquad \forall\ \gamma \in \Gamma \tag{2.8}$$

where $\chi_\gamma \in \mathrm{U}(1)$ since U_γ is a unitary operator. It is possible to show that also $\chi : \Gamma \to \mathrm{U}(1)$ that maps $\gamma \mapsto \chi_\gamma$ is a unitary representation of Γ; this map is in fact a unitary character. Since $\Gamma \simeq \mathbb{Z}^d$ we must have

$$\chi_\gamma = e^{ik\cdot\gamma}, \qquad k \in \mathbb{R}^d/\Gamma^*, \tag{2.9}$$

where Γ^* is the reciprocal lattice

$$\Gamma^* := \{\lambda \in \mathbb{R}^d \mid \lambda\cdot\gamma \in 2\pi\mathbb{Z} \quad \forall\ \gamma \in \Gamma\} \simeq (2\pi\mathbb{Z})^d. \tag{2.10}$$

Therefore

$$\Gamma^* = \mathrm{Span}_{2\pi\mathbb{Z}}(b_1, ..., b_d), \tag{2.11}$$

where $(b_1, ..., b_d)$ is the dual basis defined by $a_i \cdot b_j = \delta_{i,j}\ i, j \in \{1, ..., d\}$.
The quotient $\mathbb{R}^d/\Gamma^* \simeq \mathbb{T}^d$ is know as the Brillouin torus in condensed matter literature after the pioneering work [28]. The parameter $k \in \mathbb{T}^d$, which replaces the

linear momentum[1], is called Bloch momentum or quasi-momentum. We note that, writing $k \in \mathbb{R}^d/\Gamma^*$ simply means that $k \in \mathbb{R}^d$ is determined up to elements in Γ^*, or equivalently that χ_γ is Γ^*-periodic, i.e. $e^{i(k+\lambda)\cdot x} = e^{ik\cdot x}$ for all $\gamma \in \Gamma^*$.
Essentially, we can now see how the Hamiltonian H acts on wavefunctions $\psi_k(x)$ with well-defined quasi-momentum k; moreover since the Hamiltonian H and the translation operators U_γ commute, it will map such a wavefunction to one with the same quasi-momentum. Therefore, we can formulate the energy eigenvalue problem for fixed $k \in \mathbb{T}^d$

$$(H_k(\psi_k))(x) = E_k\psi_k(x), \tag{2.12}$$

with the additional condition

$$(U_\gamma(\psi_k))(x) = \chi_\gamma\psi_k(x) = e^{ik\cdot\gamma}\psi_k(x). \tag{2.13}$$

We give the following useful definitions.

Definition 2.1.1 (Bloch notions)**.** *Referring to* (2.12) *and* (2.13)*, the generalized eigenfunction ψ_k are known as Bloch function and the corresponding energies E_k, viewed as functions of k, are called Bloch bands. Finally, the operator H_k, which acts as the Hamiltonian but only on wavefunctions labeled by the quasi-momentum k, is called the Bloch Hamiltonian.*

Due to the k-dependent pseudo-periodic boundary conditions (2.13), Bloch eigenfunctions can be normalized in the L^2-sense not on the whole space but rather on a fundamental cell, also called the Wigner-Seitz cell, defined as

$$\mathbb{WS} := \left\{ y = \sum_{i=1}^{d} y_i a_i \mid |y_i| \leq \frac{1}{2} \right\} \simeq \left[-\frac{1}{2}, \frac{1}{2} \right]^d; \tag{2.14}$$

so

$$\int_{\mathbb{WS}} |\psi_k(y)|^2 dy = 1. \tag{2.15}$$

The solutions of the Schrödinger equation are caracterized by the following result.

Theorem 2.1.1 (Bloch Theorem)**.** *Solutions to the Schrödinger equation* (2.12) *associated to a periodic Hamiltonian H subject to the k-dependent pseudo-periodicity condition* (2.13) *can be expressed as plane waves modulated by periodic functions, i.e. $\psi_k(x) = e^{ik\cdot x}u_k(x)$ with $(U_\gamma u_k)(x) = u_k(x)$.*

Proof. Since $\psi_k(x)$ is a eigenfunction for all translation operators U_γ, by defining $u_k(x) := e^{-ik\cdot x}\psi_k(x)$ we have

$$(U_\gamma(u_k))(x) = u_k(x+\gamma) = e^{-ik\cdot(x+\gamma)}\psi_k(x+\gamma) = e^{-ik\cdot(x+\gamma)}e^{ik\cdot\gamma}\psi_k(x) = u_k(x).$$

$\square$

[1]The linear momentum is a good quantum number in the case of a quantum system in which the translational symmetry group is the full $(\mathbb{R}^d, +)$.

Therefore, translation invariance reduces to Γ-periodicity if the translations are unitarily represented in the standard way. Moreover, translation invariant functions constitute a k-independent Hilbert space, called the fiber Hilbert space

$$\mathcal{H}_k := \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \mid (U_\gamma(u))(x) = u(x),\ \gamma \in \Gamma, x \in \mathbb{R}^d\}, \tag{2.16}$$

endowed with the scalar product

$$\langle u_1 | u_2 \rangle_{\mathcal{H}_k} := \int_{\mathbb{WS}} \overline{u_1(y)} u_2(y) dy. \tag{2.17}$$

2.2 Bundles from condensed matter systems

In the case of a periodic Hamiltonian H, we have to make use of the so-called Zak-Bloch-Floquet transform, insted of the Fourier transform, to get a "partial diagonalization".

2.2.1 Zak-Bloch-Floquet transform

We start introducing the Zak-Bloch-Floquet transform

$$(U_{\mathrm{ZBF}}(\psi_k))(x) := \sum_{\gamma \in \Gamma} e^{-ik\cdot(x+\gamma)} (U_\gamma(\psi_k))(x), \qquad k \in \mathbb{R}^d, x \in \mathbb{R}^d; \tag{2.18}$$

the above series is convergent provided ψ_k decays at infinity sufficiently fast, so U_{ZBF} is defined only on a dense subspace $\mathrm{Dom}(U_{\mathrm{ZBF}}) \subset L^2(\mathbb{R}^d)$. We can immediately read the periodicity properties

$$(U_{\mathrm{ZBF}}(\psi_k))(x+\gamma) = (U_{\mathrm{ZBF}}(\psi_k))(x) \qquad \forall\ \gamma \in \Gamma; \tag{2.19a}$$

$$(U_{\mathrm{ZBF}}(\psi_{k+\lambda}))(x) = (\tau_\lambda(U_{\mathrm{ZBF}}(\psi_k)))(x) =: e^{-i\lambda\cdot x}(U_{\mathrm{ZBF}}(\psi_k))(x) \qquad \forall\ \lambda \in \Gamma^*. \tag{2.19b}$$

Condition (2.19b) is known as τ-equivariance. In view of the above conditions, k should be considered to be not in the Brillouin torus $\mathbb{T}^d$, but rather in the Brillouin zone $\mathbb{B}$

$$\mathbb{B} := \left\{ k = \sum_{i=1}^{d} k_i b_i \mid |k_i| \le \pi \right\} \simeq [-\pi, \pi]^d, \tag{2.20}$$

which is a fundamental cell for the reciprocal lattice Γ^* that is endowed with the Lebesgue measure dk inherited from the full momentum space and it is a measure space of finite measure $|\mathbb{B}|$. The following theorem expresses the properties of the Zak-Bloch-Floquet transform.

Theorem 2.2.1 (Zak-Bloch-Floquet partial diagonalization)**.** *The Zak-Bloch-Floquet transform defined in* (2.18) *extends to a unitary transformation of* $L^2(\mathbb{R}^d)$ *into the Hilbert space of* $\mathcal{H}_k$*-valued* τ*-equivariant functions*

$$\mathcal{U}_{\mathrm{ZBF}} : L^2(\mathbb{R}^d) \to \mathcal{H}_\tau$$

where

$$\mathcal{H}_\tau := \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^d; \mathcal{H}_k) \mid u_{k+\lambda}(x) = e^{-i\lambda\cdot x} u_k(x),\ \forall\ k \in \mathbb{R}^d, x \in \mathbb{R}^d, \lambda \in \Gamma^*\}$$

endowed with the scalar product

$$\langle u_1|u_2\rangle_{\mathcal{H}_\tau} := \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}}\langle u_1^{(k)}|u_2^{(k)}\rangle_{\mathcal{H}_k} dk = \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}}\int_{\mathbb{WS}} \overline{u_1^{(k)}(y)} u_2^{(k)}(y) dk dy. \tag{2.21}$$

The inverse is given by

$$(\mathcal{U}_{\mathrm{ZBF}}^{-1}(u))(x) = \frac{1}{|\mathbb{B}|}\int e^{ik\cdot x} u_k(x) dk.$$

and it is known as the Wannier function corresponding to a Bloch function u_k.

Proof. From the proof of Theorem 2.1.1, the definitions (2.16) and (2.18) and the properties (2.19a) and (2.19b) we immediately see that the Zak-Bloch-Floquet transform of ψ_k defines an $\mathcal{H}_k$-valued τ-equivariant function when ψ_k decays at infinity sufficiently fast. Therefore

$$U_{\mathrm{ZBF}} : \mathrm{Dom}(U_{\mathrm{ZBF}}) \subset L^2(\mathbb{R}^d) \to \mathrm{Range}(U_{\mathrm{ZBF}}) \subset \mathcal{H}_\tau.$$

To conclude that U_{ZBF} extends to a unitary transformation we need the bounded linear transformation theorem whose proof can be found in [4].

Theorem 2.2.2 (Bounded linear transformation)**.** *Suppose that Z is a normed space, Y is a Banach space, and $S \subset Z$ a dense linear subspace. If $T : S \to Y$ is a bounded linear transformation, then T has a unique linear extension.*

Hence, it is sufficient to show that:

1. U_{ZBF} is isometric, namely that for $\psi_k \in \mathrm{Dom}(U_{\mathrm{ZBF}})$

$$||U_{\mathrm{ZBF}}(\psi_k)||_{\mathcal{H}_\tau} = ||\psi_k||_{L^2(\mathbb{R}^d)},$$

   that shows it is a bounded operator;

2. U_{ZBF} and U_{ZBF}^{-1} are indeed inverses of one another; this together with the previous point yields in particular that the inverse transform is also isometric;

3. U_{ZBF} and U_{ZBF}^{-1} are adjoint of one another, namely for $\psi_k \in \mathrm{Dom}(U_{\mathrm{ZBF}})$ and $u \in \mathrm{Range}(U_{\mathrm{ZBF}})$ we have

$$\langle U_{\mathrm{ZBF}}(\psi_k)|u\rangle_{\mathcal{H}_\tau} = \langle \psi_k|U_{\mathrm{ZBF}}^{-1}(u)\rangle_{L^2(\mathbb{R}^d)},$$

   that shows it is unitary and so the extension will be so.

To show the above points is nothing but a matter of computations [5, 6]. □

2.2.2 Fibered Hamiltonians and the band-gap theory

The Hilbert space which constitutes the range of the Zak-Bloch-Floquet transform $\mathcal{U}_{\mathrm{ZBF}}$ is isomorphic to the constant fiber direct integral Hilbert space $\int_{\mathbb{B}}^{\oplus} \mathcal{H}_k dk$ parameterized by the measure space given by $(\mathbb{B}, \Omega, dk)$. We note that the fiber Hilbert space $\mathcal{H}_k$ does not depend on k so the direct integral Hilbert space is in fact a constant fiber direct integral Hilbert space. A particular class of operators on this Hilbert space are the so-called fibered operators.

Definition 2.2.1 (Fibered operator)**.** *A fibered operator is an operator of the form*

$$O = \int_{\mathbb{B}}^{\oplus} O_k dk,$$

where O_k is a family of operators on $\mathcal{H}_k$ parametrized by $k \in \mathbb{R}^d$ which acts fiberwise on the direct integral and such that the map $\Phi : k \mapsto O_k$ is measurable in the sense of the measurability structure of Definition 1.2.7 and Definition 1.2.9, i.e. if the functions

$$\Phi_{ij} : k \mapsto \langle e_i(k), O_k e_j(k) \rangle$$

are measurable for every i, j.

The following Proposition shows that the periodic Schrödinger operator H is fibered operator.

Proposition 2.2.1. *Let O be an operator on $L^2(\mathbb{R}^d)$ such that $[U_\gamma, O] = 0$. Then the transformed operator $\mathcal{U}_{\mathrm{ZBF}} O \mathcal{U}_{\mathrm{ZBF}}^{-1}$ is a fibered operator. Moreover the fiber operators $\{O_k\}_{k \in \mathbb{R}^d}$ fulfills τ-covariance*

$$O_{k+\lambda} = \tau_\lambda O_k \tau_\lambda^{-1}, \qquad \lambda \in \Gamma^*.$$

Proof. Due to periodicity, the action of O on $L^2(\mathbb{R}^d)$ can be restricted to functions in the k-dependent Hilbert space

$$L^2_k(\mathbb{R}^d) := \{\psi_k \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \mid U_\gamma(\psi_k) = e^{ik\cdot\gamma}\psi_k, \ \forall \gamma \in \Gamma\}.$$

We will denote this restriction as $O(k)$ and we define

$$O_k := e^{-ik\cdot x} O(k) e^{ik\cdot x},$$

that acts on $\mathcal{H}_k$. We now compute that

$$\begin{aligned}
(\mathcal{U}_{\mathrm{ZBF}}(O(\psi_k)))(x) &= \sum_{\gamma\in\Gamma} e^{-ik\cdot(x+\gamma)} (U_\gamma(O(\psi_k)))(x) = \\
&= \sum_{\gamma\in\Gamma} e^{-ik\cdot(x+\gamma)} (O(U_\gamma(\psi_k)))(x) = \\
&= O_k \left(\sum_{\gamma\in\Gamma} e^{-ik\cdot(x+\gamma)} (U_\gamma(\psi_k))(x) \right) = (O_k((\mathcal{U}_{\mathrm{ZBF}}(\psi_k))))(x);
\end{aligned} \tag{2.22}$$

hence $O_k = \mathcal{U}_{\mathrm{ZBF}} O \mathcal{U}_{\mathrm{ZBF}}^{-1}\Big|_{L^2_k(\mathbb{R}^d)}$. Moreover, if $\psi_{k+\lambda} \in L^2_{k+\lambda}(\mathbb{R}^d)$ with $\lambda \in \Gamma^*$, we have

$$U_\gamma(\psi_{k+\lambda}) = e^{i(k+\lambda)\cdot\gamma}\psi_{k+\lambda} = e^{ik\cdot\gamma}\psi_{k+\lambda}, \qquad \gamma \in \Gamma;$$

therefore $L^2_{k+\lambda}(\mathbb{R}^d) = L^2_k(\mathbb{R}^d)$ and so $O(k+\lambda) = O(k)$. In the end, for $\lambda \in \Gamma^*$, we have

$$O_{k+\lambda} = e^{-i(k+\lambda)\cdot x} O(k+\lambda) e^{i(k+\lambda)\cdot x} = \tau_\lambda e^{-ik\cdot x} O(k) e^{ik\cdot x} \tau_\lambda^{-1} = \tau_\lambda O_k \tau_\lambda^{-1}.$$

□

An immediate application of the above Proposition is that

$$\mathcal{U}_{\mathrm{ZBF}} H \mathcal{U}_{\mathrm{ZBF}}^{-1} = \int_{\mathbb{B}}^{\oplus} H_k dk, \tag{2.23}$$

where $H_k := -\frac{1}{2m}(-i\nabla + k)^2 + V_\Gamma(x)$ is the fiber Hamiltonian. Assuming that V_Γ is relatively bounded with respect to $-\Delta$ with vanishing relative bound, the fiber Hamiltonians $\{H_k\}_{k\in\mathbb{R}^d}$ define a family of self-adjoint operators which is entire analytic in the sense of Kato with compact resolvent. This means that if we take $k \in \mathbb{C}^d$, the fiber Hamiltonians $\{H_k\}_{k\in\mathbb{C}^d}$ are such that:

1. the domain $\mathrm{Dom}(H_k) \subset \mathcal{H}_k$ is independent of $k \in \mathbb{C}^d$;
2. the set $R := \{(k, z) \in \mathbb{C}^d \times \mathbb{C} \mid z \in \rho(H_k)\}$ is open and the resolvent map $\varrho : R \to \mathcal{B}(\mathcal{H}_k)$ that maps $(k, z) \mapsto (H_k - z1_{\mathcal{H}_k})^{-1}$ is analytic on R, with values in the algebra of compact operators.

Under these hypotheses, fiber Hamiltonians H_k are self-adjoint on a k-independent domain contained into $\mathcal{H}_k$, are bounded from below uniformly in k and each H_k has compact resolvent with discrete spectrum accumulating at infinity. Hence for $n \in \mathbb{N}$, there exists $u_{n,k} \in \mathcal{H}_k$ normalized to one such that

$$(H_k(u_{n,k}))(x) = E_{n,k} u_{n,k}(x). \tag{2.24}$$

This is a spectral theorem for a periodic Schrödinger operator H and it tells us that the spectrum of H is labeled by a discrete index, n, and a continuous one, k. The $E_{n,k}$ are assumed to be labeled in non-decreasing order

$$E_{0,k} \leq E_{1,k} \leq ... \leq E_{s,k} \leq E_{s+1,k} \leq ... \leq +\infty, \tag{2.25}$$

Therefore, we could draw the spectrum of H as a numerable set of function graphs as k varies in the Brillouin zone. These $E_{n,k}$ are called energy bands or Bloch bands. They are Γ^*-periodic, i.e. $E_{n,k+\lambda} = E_{n,k}$, as a result of the τ-covariance property of the fiber Hamiltonians

$$H_{k+\lambda} = \tau_\lambda H_k \tau_\lambda^{-1}. \tag{2.26}$$

The total spectrum of H can be reconstructed as the union of the Bloch bands

$$\sigma(H) = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{B}} \{E_{n,k}\}. \tag{2.27}$$

The image of the Γ^*-periodic function $E_{n,k} : \mathbb{R}^d \to \mathbb{R}$ constitutes a part of the spectrum of H. Different Bloch bands may have overlapping images, and their union in the spectrum of H is referred to as a spectral band. When these spectral bands do not overlap, they are said to be separated by a spectral gap. As a result, the spectrum of H alternates between spectral bands and spectral gaps, forming what is known as the band-gap structure. From a physical point of view, band-gap structure theory has been successfully used to explain many physical properties of solids, such as electrical resistivity, electrical conductivity, optical absorption and it is the foundation of the understanding of all solid-state devices like transistors or solar cells. An example of energy bands for different solids useful for solid-state devices is reported in the picture below [29]. Graphene, Silicene and Germanene are allotropic states, respectively, of Carbon, Silicon and Germanium consisting of a single layer of atoms arranged in a hexagonal lattice. On the one hand, Graphene has exceptional properties: it has the theoretical strength of diamond and the flexibility of plastic, as well as remarkable electrical and thermal conductivity. On the other hand, Silicene and Germanene are possible alternatives for improving the performance and scalability of semiconductor electronics.

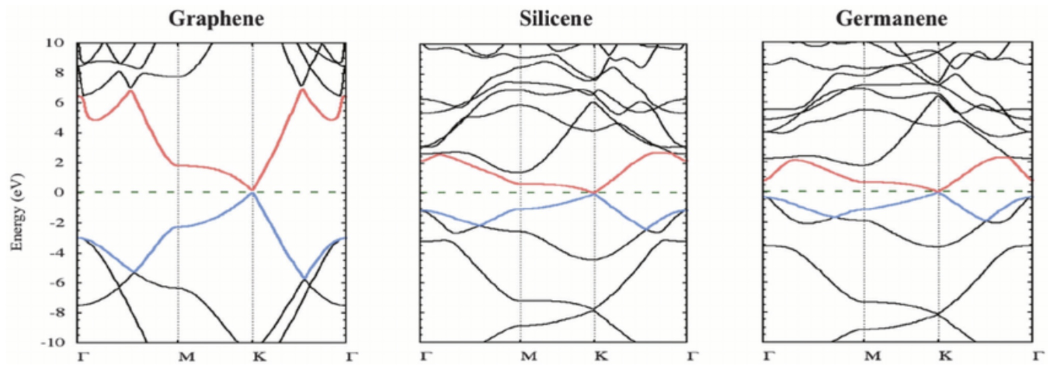

Figure 2.1. Band structures of Graphene (left), Silicene (centre) and Germanene (right) in appropriate units. Valence and conduction bands are highlighted in blue and red respectively. The Fermi levels (horizontal green dashed lines) have been shifted to 0 eV [29]. The letters mark some highly symmetric points in the Brillouin zone.

We now assume that the first $m < n$ Bloch bands are separated from the rest of the spectrum by a spectral gap, so let us define

$$\sigma_{\mathrm{occ}}(H) := \bigcup_{i \leq m} \bigcup_{k \in \mathbb{B}} \{E_{i,k}\} \tag{2.28}$$

and we further assume that points in $\sigma_{\mathrm{occ}}(H)$ are well separated from points in $\sigma_{\mathrm{unocc}}(H) := \sigma(H) \setminus \sigma_{\mathrm{occ}}(H)$, i.e.

$$\mu_{\mathrm{occ}} < \mu_{\mathrm{F}} < \mu, \qquad \mu_{\mathrm{occ}} \in \sigma_{\mathrm{occ}}(H),\ \mu \in \sigma_{\mathrm{unocc}}(H),\ \mu_{\mathrm{F}} \in \mathbb{R}. \tag{2.29}$$

Physically, when a large number of electrons are added to the system and occupy the first m energy levels (the occupied bands), the Fermi energy μ_{F} can fall within a spectral gap. This means that it costs energy for an electron to move from the occupied bands (valence bands) to the unoccupied bands (conduction bands). Under these conditions we are describing an insulating material or a semiconductor if the Fermi energy is very slightly outside the valence band[2]; in contrast, a semi-metal or a metal is formed when the Fermi energy lies within a spectral band as Figure 2.2 shows [30]. The crucial point is that topological phases and topological transport

[2]So electrons can jump to conduction bands only thanks to thermal excitations. Specifically, if the band gap between valence and conduction bands is more or less one hundredth of an electronvolt thermal excitation at room temperature is sufficient. At present, no semiconductors with such a small band gap are known, but there are semiconductors with band gaps of about 0.12 eV, which corresponds to about 1200° C.

properties of the system are contained in the set of occupied energy states, spanned by the corresponding Bloch functions.

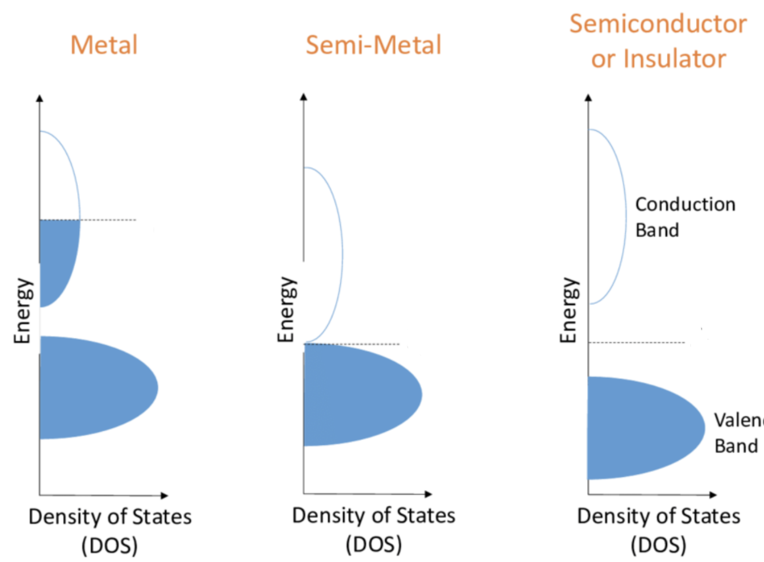

Figure 2.2. A comparison of typical metal, semi-metal, and semiconductor/insulator. Dotted lines represent the Fermi energy level in each case [30].

2.2.3 The Bloch bundle

To deal with the part of the spectrum given by $\sigma_{\text{occ}}(H)$ and the corresponding Bloch functions, it is useful to define the spectral eigenprojections by functional calculus as

$$P_k := \chi(H_k < \mu_{\text{F}}), \tag{2.30}$$

where $\chi(\text{condition})$ is the indicator function of a condition. Therefore, P_k projects onto the subspace of $\mathcal{H}_k$ spanned by the Bloch functions associated to these occupied energy levels, in particular $P_k = P_k^* = P_k^2$ since $\chi(\mu < \mu_{\text{F}})$ is a real-valued function such that $\chi(\mu < \mu_{\text{F}})^2 = \chi(\mu < \mu_{\text{F}})$. Moreover, it is possible to show that the family of spectral eigenprojections of H can be computed by the Riesz formula

$$P_k = \frac{i}{2\pi} \oint_\alpha (H_k - z 1_{\mathcal{H}_k})^{-1} \, dz, \qquad k \in \mathbb{T}^d, \tag{2.31}$$

where α is a simple, positively oriented, closed countour in $\mathbb{C}$ intersecting the real axis at μ_0 and μ_{F}, where μ_0 is a lower bound for the spectrum of H.

Proposition 2.2.2 (Properties of the spectral eigenprojections)**.** *P_k depends analytically on $k \in \mathbb{R}^d$ and fulfills τ-covariance*

$$P_{k+\lambda} = \tau_\lambda P_k \tau_\lambda^{-1}, \qquad \lambda \in \Gamma^*.$$

Proof. The analyticity of P_k as a function of k follows from the analyticity of the resolvent map for H_k, together with the fact that the integration contour α is compact. The τ-covariance is also inherited from H_k and its resolvent by functional calculus. □

Recall that for $\mathcal{H}$ a complex Hilbert space, the m-dimensional subspaces of $\mathcal{H}$ are in one-to-one correspondence with the elements of the Grassmannian $\mathrm{Gr}_{m,\infty}(\mathcal{H})$,

$$\mathrm{Gr}_{m,\infty}(\mathcal{H}) := \left\{ P_k \in \mathcal{B}(\mathcal{H}) \mid P_k^2 = P_k = P_k^*,\ \mathrm{tr}(P_k) = m \right\},$$

which is comprised of rank m orthogonal projections in the algebra $\mathcal{B}(\mathcal{H})$ of linear and bounded operators on $\mathcal{H}$. We also note that the spectral eigenprojections P_k constitute the Zak-Bloch-Floquet fibers of the spectral eigenprojection

$$\Pi := \chi(\mu < \mu_{\mathrm{F}}) = \frac{i}{2\pi} \int_\alpha \left(H - z 1_{L^2(\mathbb{R}^d)} \right)^{-1} dz, \tag{2.32}$$

so

$$\mathcal{U}_{\mathrm{ZBF}} \Pi \mathcal{U}_{\mathrm{ZBF}}^{-1} = \int_{\mathbb{B}}^{\oplus} P_k dk. \tag{2.33}$$

It is crucial for our scopes to note that, while the spectral eigenprojections P_k are smooth in k, the individual Bloch functions $u_{n,k}$ are not. This is because, at certain values of k, some eigenvalues of H_k may become degenerate which can cause the corresponding eigenvectors to be ill-defined. Given this, it is often more practical to examine the normalized eigenfunctions of the eigenprojections directly, rather than those of the Hamiltonian

$$P_k(\phi_k) = \phi_k, \qquad \phi_k \in \mathcal{H}_k,\ ||\phi_k||_{\mathcal{H}_k} = 1; \tag{2.34}$$

where ϕ_k are known as quasi-Bloch functions. A set of regular quasi-Bloch functions, possibly compatible with τ-equivariance, that span the ranges of the spectral eigenprojections orthonormally, would provide a convenient basis for the portion of $\mathcal{H}_k$ corresponding to the occupied energy levels. However, it turns out that the existence of such a regular, τ-equivariant, orthonormal family of quasi-Bloch functions can be topologically obstructed. In the end, we have constructed a family of m-dimensional subspaces of $\mathcal{H}_k$, given by the ranges of the spectral eigenprojections, which vary smoothly with k in the Brillouin zone. In fact, since the first m bands are associated to m Bloch functions, these are m-dimensional subspaces. This information can be recasted into a vector bundle over the Brillouin torus $\mathbb{T}^d$ as we are going to see shortly.
Before moving on, let us recall the definition of a vector bundle.

Definition 2.2.2 (Vector bundle)**.** *Let $\mathbb{K}$ be a field. A $\mathbb{K}$-vector bundle of rank m over a topological space X is a triple $\mathcal{E} = (E, \pi, X)$ where:*

1. *E is the total space of the bundle;*
2. *X is the base space of the bundle;*
3. *$\pi : E \to X$ is a surjective map, called the projection;*

such that:

1. *for each $x \in X$, the fiber over x, denoted $E_x := \pi^{-1}(\{x\})$, is a vector space over $\mathbb{K}$;*

2. *for each $x \in X$, there exists an open neighborhood $U \subset X$ containing x, an integer $m_x \in \mathbb{N}$ and a homeomorphism $\varphi_U : U \times \mathbb{K}^{m_x} \to \pi^{-1}(U)$, called a local trivialization. The map φ is fiberwise linear, i.e. for each $x \in U$ the restriction of φ to $\{x\} \times \mathbb{K}_x^{m_x}$ is a linear map*
$$\varphi_x : \{x\} \times \mathbb{K}_x^{m_x} \to E_x, \quad (x, v) \mapsto \varphi(x, v),$$
which is a isomorphism from the fiber $\mathbb{K}_x^{m_x}$ to the fiber E_x;

3. *the map $\Phi : x \mapsto m_x$ is locally constant, and hence constant on connected components of X. It represents the dimension of each fiber as a vector space and is called the rank of the bundle uniquely defined on connected X;*

4. *on a pair of trivializing neighborhoods U and V, such that $U \cap V \neq \emptyset$, with local trivializations*
$$\varphi_U : U \times \mathbb{K}^m \to \pi^{-1}(U) \quad \text{and} \quad \varphi_V : V \times \mathbb{K}^m \to \pi^{-1}(V)$$
there exists a a well-behaved map
$$\varphi_U^{-1} \circ \varphi_V : (U \cap V) \times \mathbb{K}^m \to (U \cap V) \times \mathbb{K}^m,$$
defined on the overlap $U \cap V$, that satisfies
$$\varphi_U^{-1} \circ \varphi_V(x, v) = (x, g_{UV}(x)(v))$$
for some function $g_{UV} : U \cap V \to \mathrm{GL}(m, \mathbb{K})$, which maps $x \mapsto g_{UV}(x)$, satisfying the Čech 1-cocycle condition.

The above definition is given in the category of topological spaces but can be stated also in differential, smooth or holomorphic contexts. Furthermore, the concept of a vector bundle can be naturally extended to bundles whose fibers are modules over a ring, possibly non-commutative. This framework accommodates important examples, such as bundles with fibers given by left or right modules over the division ring of quaternions $\mathbb{H}$, which appear in contexts ranging from non-commutative geometry to the study of topological projective modules [31, 32, 21]. We now introduce three important cases of $\mathbb{K}$-vector bundles.

Definition 2.2.3 (Trivial bundle)**.** *The trivial bundle of rank m over X is the triple $\mathcal{E}_{\mathrm{tri}}^{(m)} = (X \times \mathbb{K}^m, \mathrm{Proj}_1, X)$ where Proj_1 is just the projection on the first factor.*

Definition 2.2.4 (Tautological bundle over the Grassmannian)**.** *Let V be a vector space on $\mathbb{K}$ and let $\mathrm{Gr}_{m,\dim(\mathrm{V})}(V)$ be the Grassmannian of m-dimensional vector subspaces in V. The tautological bundle over the Grassmannian $\mathrm{Gr}_{m,\dim(\mathrm{V})}(V)$ is the triple $\mathcal{O}_{\mathrm{Gr}_{m,\dim(\mathrm{V})}(V)} = (E_{\mathrm{tau}}, \pi_{\mathrm{tau}}, \mathrm{Gr}_{m,\dim(\mathrm{V})}(V))$ such that:*

1. *The total space E_{tau} is the set of all pairs $([W], w)$ consisting of a point $[W]$ of the Grassmannian and a vector $w \in W \subset V$*
$$E_{\mathrm{tau}} := \{([W], w) \mid [W] \in \mathrm{Gr}_{m,\dim(\mathrm{V})}(V), w \in W\},$$
with the subspace topology of the cartesian product $\mathrm{Gr}_{m,\dim(\mathrm{V})}(V) \times V$.

2. The projection π_{tau} *is given by*

$$\pi_{\text{tau}} : E_{\text{tau}} \to \mathrm{Gr}_{m,\dim(\mathrm{V})}(V), \quad ([W], w) \mapsto [W].$$

Definition 2.2.5 (Quotient bundle over the Grassmannian)**.** *Let V be a vector space on $\mathbb{K}$ and let $\mathrm{Gr}_{m,\dim(\mathrm{V})}(V)$ be the Grassmannian of m-dimensional vector subspaces in V. The quotient bundle over the Grassmannian $\mathrm{Gr}_{m,\dim(\mathrm{V})}(V)$ is the triple $\mathcal{Q}_{\mathrm{Gr}_{m,\dim(\mathrm{V})}(V)} = (E_{\text{quo}}, \pi_{\text{quo}}, \mathrm{Gr}_{m,\dim(\mathrm{V})}(V))$ such that:*

1. *The total space E_{quo} is the set of all pairs $([W], w)$ consisting of a point $[W]$ of the Grassmannian and a vector $w \perp W \subset V$*

$$E_{\text{quo}} := \{([W], w) \mid [W] \in \mathrm{Gr}_{m,\dim(\mathrm{V})}(V), w \perp W\},$$

with the subspace topology of the cartesian product $\mathrm{Gr}_{m,\dim(\mathrm{V})}(V) \times V$.

2. *The projection π_{quo} is given by*

$$\pi_{\text{quo}} : E_{\text{quo}} \to \mathrm{Gr}_{m,\dim(\mathrm{V})}(V), \quad ([W], w) \mapsto [W].$$

We are now ready to build up the Bloch bundle, i.e. the bundle associated to a condensed matter system.

Definition 2.2.6 (Bloch bundle)**.** *The Bloch bundle associated to a condensed matter system is the triple $\mathcal{E}_{\text{Bloch}} = (E_{\text{Bloch}}, \pi_{\text{Bloch}}, X_{\text{Bloch}})$ such that:*

1. *the base X_{Bloch} is the Brillouin torus $\mathbb{T}^d := \mathbb{R}^d/\Gamma^*$;*
2. *the total space E_{Bloch} is defined as*

$$E_{\text{Bloch}} := \{[k, \phi_k] \in (\mathbb{R}^d \times \mathcal{H}_k)/\sim_\lambda \ \mid \phi_k \in \mathrm{Range}(P_k)\}$$

where

$$(k, \phi_k) \sim_\lambda (k', \phi'_{k'}) \qquad \text{if} \quad k' = k + \lambda,\ \phi'_{k'} = \tau_\lambda(\phi_k),$$

for some $\lambda \in \Gamma^*$;

3. *the projection π_{Bloch} is given by*

$$\pi_{\text{Bloch}} : E_{\text{Bloch}} \to \mathbb{T}^d, \quad [k, \phi] \mapsto k \bmod \Gamma^*.$$

The next Proposition shows that the Bloch bundle is in fact a complex vector bundle of rank m.

Proposition 2.2.3. *The triple $\mathcal{E}_{\text{Bloch}} = (E_{\text{Bloch}}, \pi_{\text{Bloch}}, X_{\text{Bloch}})$ defined in Definition 2.2.6 is a complex vector bundle of rank m.*

Proof. First of all let us show that E_{Bloch} and π_{Bloch} are well defined, i.e. do not depend from the representative. Hence, we want to show that if $(k, \phi_k) \sim_\lambda (k', \phi_{k'})$ then:

1. if $P_k(\phi_k) = \phi_k$ then $P_{k'}(\phi'_{k'}) = \phi'_{k'}$;

2. $k' \simeq k \bmod \Gamma^*$.

For the first point, it is enough to note that

$$P_{k'}(\phi'_{k'}) = P_{k+\lambda}(\tau_\lambda(\phi_k)) = \tau_\lambda(P_k(\phi_k)) = \tau_\lambda(\phi_k) = \phi'_{k'}.$$

The second point is just so by definition. The fiber on a point k of the Brillouin torus is given by

$$(E_{\mathrm{Bloch}})_k = \mathrm{Range}(P_k)$$

that is naturally a complex vector space, as it is a vector subspace of $\mathcal{H}_k$. The rank of the Bloch bundle coincides then with the dimension m of the range of P_k, which in solids state physics corresponds to the number of occupied Bloch bands.
To show the local triviality we need the Kato-Nagy formula [33].

Proposition 2.2.4 (Kato-Nagy formula)**.** *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be orthogonal projections on the Hilbert space $\mathcal{H}$. Assume that $\|P - Q\| < 1$. Then the operator*

$$W := \left(1_{\mathcal{H}} - (P-Q)^2\right)^{-\frac{1}{2}} \left[PQ + (1_{\mathcal{H}} - P)(1_{\mathcal{H}} - Q)\right]$$

is unitary and intertwines P and Q, i.e.

$$P = WQW^{-1}.$$

Let us fix a $k_0 \in \mathbb{R}^d$; from the analyticity of the map $\Phi : k \mapsto P_k$, it follows that there exists an open neighbourhood $U \ni k_0$ such that

$$\|P_k - P_{k_0}\| < 1, \qquad \forall\ k \in U.$$

Let then $W_{k;k_0}$ be the Kato-Nagy unitary intertwining such that

$$P_k = W_{k;k_0} P_{k_0} W_{k;k_0}^{-1};$$

and let us choose an orthonormal basis $\{\varphi_1(k_0), \dots, \varphi_m(k_0)\}$ for $\mathrm{Range}(P_{k_0})$. A local trivialization of the Bloch bundle over U, $\varphi_U : U \times \mathbb{C}^m \to \pi^{-1}(U) = \sqcup_{k \in U} \mathrm{Range}(P_k)$, can be defined via

$$\varphi_U : (k, (v^1, \dots, v^m)) \mapsto \left(k, \sum_{\alpha=1}^{m} v^\alpha W_{k;k_0}(\varphi_\alpha(k_0))\right),$$

since

$$P_k(W_{k;k_0}(\varphi_\alpha(k_0))) = W_{k;k_0}(P_{k_0}(\varphi_\alpha(k_0))) = W_{k;k_0}(\varphi_\alpha(k_0)),$$

so that $W_{k;k_0}(\varphi_\alpha(k_0)) \in \mathrm{Range}(P_k)$ for all $\alpha \in \{1, \dots, m\}$ and these vectors are also orthonormal because $W_{k;k_0}$ is unitary, hence they are a basis. □

The above motivates our brief dive into the study of vector bundles, connections on them and their classification up to isomorphism.

2.3 Vector bundles and connections

We now enter in the study of vector bundles; we first give some definitions directly in the framework of vector bundles to then generalize and introduce principal bundles and study their connections and curvatures.

2.3.1 Frames and the Whitney sum bundle

We start recalling what a section of a bundle is. Let (E, π, X) be a fiber bundle. A section $s : X \to E$ is a smooth map which satisfies $\pi \circ s = 1_X$. Moreover, given a point $x \in X$, $s(x) = s|_x$ is an element of $E_x := \pi^{-1}(x)$. The set of global sections, i.e. sections on X, is denoted by $\Gamma(X, E)$ or $H^0(X, E)$. If $U \subset X$, we may talk of a local section which is defined only on U and $\Gamma(U, E)$ or $H^0(U, E)$ denotes the set of local sections on U. In the case the fiber has a vector space structure, so the fiber bundle is enhanced to a vector bundle, we can give the following Definition.

Definition 2.3.1 (Frame). *Let $\mathcal{E} = (E, \pi, X)$ be a $\mathbb{K}$-vector bundle whose fiber has dimension m. On a chart U_i, we have that $\pi^{-1}(U_i)$ is trivial, i.e. $\pi^{-1}(U_i) \simeq U_i \times E|_{U_i}$, and we may choose m linearly independent sections*

$$\{e_1(x), \dots, e_m(x)\}$$

over $U_i \ni x$. These sections are said to define a frame over U_i.

Given a frame over U_i, we have a natural map $E_x \to \mathbb{K}^m$ given by

$$v = \sum_{\alpha=1}^{m} v^\alpha e_\alpha(x) \longmapsto (v^1, ..., v^m) \in \mathbb{K}^m; \tag{2.35}$$

the local trivialization is

$$\varphi_{U_i}^{-1}(v) = (x, (v^1, ..., v^m)). \tag{2.36}$$

Let $U_i \cap U_j \neq \emptyset$ and consider the change of frames. Let $\{e_1(x), \dots, e_m(x)\}$ be a frame on U_i and let $\{\tilde{e}_1(x), \dots, \tilde{e}_m(x)\}$ be a frame on U_j, where $x \in U_i \cap U_j$. A vector $\tilde{e}_\beta(x)$ is expressed as

$$\tilde{e}_\beta(x) = \sum_{\alpha=1}^{m} e_\alpha(x) G(x)^\alpha_\beta \tag{2.37}$$

where $G(x)^\alpha_\beta \in \mathrm{GL}(m, \mathbb{K})$; moreover, any vector $v \in \pi^{-1}(x)$ is expressed as

$$v = \sum_{\alpha=1}^{m} v^\alpha e_\alpha(x) = \sum_{\alpha=1}^{m} \tilde{v}^\alpha \tilde{e}_\alpha(x). \tag{2.38}$$

Therefore,

$$\tilde{v}^\beta = G^{-1}(x)^\beta_\alpha v^\alpha. \tag{2.39}$$

Thus, we find that the transition function $g_{U_i U_j}(x)$ is given by the matrix $G^{-1}(x)$. Hence, we note that giving a local trivialization is equivalent to giving a local frame; if the frame is global then we have a global trivialization and the vector bundle is isomorphic to the trivial one.
We now introduce the pullback of vector bundle to state an important result that will be useful in their classification.

Definition 2.3.2 (Pullback vector bundle)**.** *Let $\mathcal{E} = (E, \pi, X)$ be a $\mathbb{K}$-vector bundle of rank m. Given a map $f : Y \to X$, we define the pullback of E by f as*

$$f^*E := \{(y, u) \in Y \times E \mid f(y) = \pi(u)\};$$

where the fiber $\mathbb{K}^m_y$ of f^*E is just a copy of the fiber $\mathbb{K}^m_{f(y)}$ of E. By defining

$$f^*E \xrightarrow{\Pi} Y \quad \text{and} \quad f^*E \xrightarrow{\mathrm{Proj}_2} E$$

with $\Pi(y, u) = y$ and $\mathrm{Proj}_2(y, u) = u$, the pullback f^*E is endowed with the structure of a $\mathbb{K}$-vector bundle of rank m, denoted $\mathcal{F}^*\mathcal{E} = (f^*E, \Pi, Y)$ such that the following diagram commutes:

$$\begin{tikzcd} f^*E \arrow[r, "\mathrm{Proj}_2"] \arrow[d, "\Pi"'] & E \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

The commutativity of the diagram follows since, by definition

$$\pi(\mathrm{Proj}_2(y, u)) = \pi(u) = f(y) = f(\Pi(y, u)) \qquad \forall\ (y, u) \in f^*E. \tag{2.40}$$

In particular, if $Y = X$ and $f = 1_X$, then the two vector bundles $\mathcal{F}^*\mathcal{E}$ and $\mathcal{E}$ are isomorphic one to the other. Moreover, if $\{U_i\}$ is an open covering of X and $\{\varphi_{U_i}\}$ a corresponding set of local trivializations of (E, π, X), then $\{f^{-1}(U_i)\}$ defines an open covering of Y, such that $\mathcal{F}^*\mathcal{E}$ is locally trivial. Furthermore, if $g_{U_iU_j}(x)$ is a transition function in x for (E, π, X) then the transition function in $y = f^{-1}(x)$ for (f^*E, Π, Y) is

$$g_{f^{-1}(U_i)f^{-1}(U_j)}(y) = g_{U_iU_j}(x). \tag{2.41}$$

Given two vector bundles with ranks m_1 and m_2 there are two important constructions we need to develop in view of more advanced material.

Definition 2.3.3 (Product bundle)**.** *Let $\mathcal{E} = (E, \pi, X)$ and $\mathcal{E}' = (E', \pi', X')$ be $\mathbb{K}$-vector bundles with typical fibers $\mathbb{K}^{m_1}$ and $\mathbb{K}^{m_2}$, respectively. The product bundle is the triple*

$$\mathcal{E} \times \mathcal{E}' = (E \times E', \pi \times \pi', X \times X');$$

it is a $\mathbb{K}$-vector bundle whose typical fiber is $\mathbb{K}^{m_1} \oplus \mathbb{K}^{m_2}$.

A particular relevant construction that melts the pullback and product bundles is the following.

Definition 2.3.4 (Whitney sum bundle)**.** *Let $\mathcal{E} = (E, \pi, X)$ and $\mathcal{E}' = (E', \pi', X)$ be $\mathbb{K}$-vector bundles of ranks m_1 and m_2. The Whitney sum bundle $\mathcal{E} \oplus \mathcal{E}'$ is a pullback bundle of $\mathcal{E} \times \mathcal{E}'$ via the map $f : X \to X \times X$ defined by $f(x) = (x, x)$, i.e.*

$$\begin{tikzcd} E \oplus E' \arrow[r, "\mathrm{Proj}_2"] \arrow[d, "\Pi"'] & E \times E' \arrow[d, "\pi \times \pi'"] \\ X \arrow[r, "f"'] & X \times X \end{tikzcd}$$

Thus

$$E \oplus E' = \{(x, (u, u')) \in X \times E \times E' \mid (\pi \times \pi')(u, u') = f(x) \text{ for some } x \in X\}.$$

In short, $(E \oplus E', \Pi, X)$ is a vector bundle over X whose fiber at $x \in X$ is $\mathbb{K}_x^{m_1} \oplus \mathbb{K}_x^{m_2}$. Furthermore, let $\{U_i\}$ be an open covering of X, and let $g^E_{U_iU_j}(x)$ and $g^{E'}_{U_iU_j}(x)$ be the transition functions of (E, π, X) and (E', π', X) at $x \in X$, respectively. Then the transition function $G_{U_iU_j}$ of $\mathcal{E} \oplus \mathcal{E}'$ in $x \in X$ is a $(m_1 + m_2) \times (m_1 + m_2)$ matrix given by

$$G_{U_iU_j}(x) = \begin{pmatrix} g^E_{U_iU_j}(x) & 0 \\ 0 & g^{E'}_{U_iU_j}(x) \end{pmatrix} \tag{2.42}$$

which acts on from the left.
Pullback constructions, product bundles and Whitney sum bundles play a central role in the proof of key results such as the Mayer-Vietoris sequence in bundles cohomology, the splitting principle in the theory of characteristic classes, and the Grothendieck-Riemann-Roch theorem which extends Hirzebruch-Riemann-Roch theorem to more general contexts [34, 35].

2.3.2 Principal bundles and connections

We now generalize our framework from vector bundles to principal bundles. Informally speaking, a principal G-bundle over X, indicated as $P(X, \mathrm{G})$, is a fiber bundle (P, π, X) such that the fiber is isomorphic to the group G. The transition function acts on the fiber on the left as before but, in addition, we may also define the action of $(\mathrm{G}, \cdot)$ on the fibers on the right. Let $\{U_i\}$ be an open covering of X and $\varphi_{U_i} : U_i \times \mathrm{G} \to \pi^{-1}(U_i)$ be the local trivialization given by $\varphi^{-1}_{U_i}(u) = (x, g_i)$, where $u \in \pi^{-1}(U_i)$ and $x = \pi(u)$. The right action of $(\mathrm{G}, \cdot)$ on $\pi^{-1}(U_i)$ is defined by

$$\varphi^{-1}_{U_i}(uh) := (x, g_i \cdot h), \tag{2.43}$$

for any $h \in \mathrm{G}$ and $u \in \pi^{-1}(x)$. Since the right action commutes with the left action, this definition is independent of the local trivializations. In fact, if $x \in U_i \cap U_j$,

$$uh = \varphi_{U_j}(x, g_j \cdot h) = \varphi_{U_j}(x, g_{U_iU_j}(x)(g_i) \cdot h) = \varphi_{U_i}(x, g_i \cdot h).$$

Therefore, the right multiplication is defined without reference to the local trivializations as

$$R_g : P \times \mathrm{G} \to P, \quad (u, h) \mapsto uh; \tag{2.44}$$

we note that since the map π projects on the first component, we have

$$\pi(uh) = \pi(u) = x. \tag{2.45}$$

The right action of $(\mathrm{G}, \cdot)$ on the fiber at the point $x \in X$, $\pi^{-1}(x)$, is transitive since $(\mathrm{G}, \cdot)$ acts on itself transitively on the right and the fiber is isomorphic to G. Thus, for any $u_1, u_2 \in \pi^{-1}(x)$, there exists an element $h \in \mathrm{G}$ such that $u_1 = u_2 h$. Then, if $\pi(u) = x$, we can construct the whole fiber as

$$\pi^{-1}(x) = \{uh \mid h \in \mathrm{G}\}.$$

The action is also free: if $uh = u$ for some $u \in P$ then h must be the unit element e of G. Indeed, if $u = \varphi_{U_i}(x, g_i)$ then consider uh; on the one hand

$$uh = \varphi_{U_i}(x, g_i)h = u = \varphi_{U_i}(x, g_i), \tag{2.46}$$

while, on the other hand, by definition

$$uh = \varphi_{U_i}(x, g_i \cdot h). \tag{2.47}$$

Since φ_{U_i} is bijective, we must have $g_i \cdot h = g_i$ hence $h = e$. We can condensate all in the following Definition.

Definition 2.3.5 (Principal G-bundle)**.** *Let* G *be a topological group. A principal* G*-bundle, $P(X, \mathrm{G})$, is a fiber bundle (P, π, X) whose fiber is isomorphic to* G *together with a continuous right action*

$$P \times \mathrm{G} \to P, \quad (u, h) \mapsto uh,$$

with $h \in \mathrm{G}$, which preserves the fibers and acts freely and transitively on them.

We can enhance this definition by requiring that the right action is, for example, smooth and $(\mathrm{G}, \cdot)$ is a Lie group. An example of principal G-bundle is the Hopf fibration where $P = S^3, \mathrm{G} = \mathrm{U}(1), X = S^2$.
In this framework, vector bundles are recovered as particular associated bundles to a principal G-bundle. In general, given a principal G-bundle $P(X, \mathrm{G})$, we may construct an associated fiber bundle.

Definition 2.3.6 (Associated fiber bundle)**.** *Let $\rho : \mathrm{G} \to \mathrm{Hom}(Y)$ be a continuous left action on a topological manifold Y and $P(X, \mathrm{G})$ a principal* G*-bundle with projection map π. Define an action of $g \in \mathrm{G}$ on $P \times Y$ by*

$$(u, y) \mapsto (ug, \rho(g^{-1})y),$$

where $u \in P$ and $y \in Y$. Then, the associated fiber bundle over X is the quotient space

$$P \times_\rho Y \ := (P \times Y)/\mathrm{G},$$

in which the points (u, y) and $(ug, \rho(g^{-1})y)$ are identified together with the projection

$$\pi_\rho : P \times_\rho Y \to X, \qquad \pi_\rho([u, y]) = \pi(u).$$

Let us now consider the case in which Y is a m-dimensional vector space V on $\mathbb{K}$ and let ρ be a m-dimensional representation of G on V. The associated $\mathbb{K}$-vector bundle $P \times_\rho V$ is defined by identifying the points (u, v) and $(ug, \rho(g)^{-1}v)$ of $P \times V$ where $u \in P$, $g \in \mathrm{G}$ and $v \in V$. For example, associated with the principal $\mathrm{GL}(m, \mathbb{K})$-bundle $P(X, \mathrm{GL}(m, \mathbb{K}))$ is a $\mathbb{K}$-vector bundle over X with typical fiber $\mathbb{K}^m$. Conversely, a $\mathbb{K}$-vector bundle naturally induces a principal $\mathrm{GL}(m, \mathbb{K})$-bundle associated with which it is: let (E, π, X) be a $\mathbb{K}$-vector bundle with typical fiber $\mathbb{K}^m$, then we have a principal bundle $P(X, \mathrm{G})$ over X, called the frame bundle, by employing the same transition functions and the structure group $\mathrm{G} = \mathrm{GL}(m, \mathbb{K})$. If $g_{U_iU_j}(x)$ is a transition function for our $\mathrm{GL}(m, \mathbb{K})$-bundle then $\rho(g_{U_iU_j}(x))$ is a transition function for the associated $\mathbb{K}$-vector bundle.

Let u be an element of a principal G-bundle $P(X, \mathrm{G})$ and fix $x = \pi(u)$. The vertical subspace V_uP is a subspace of the tangent space T_uP which is tangent in the point u to the fiber at point $x \in X$. Concretely, V_uP is constructed taking an element $\mathfrak{X} \in (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, where $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is the Lie algebra of a Lie group $(\mathrm{G}, \cdot)$, and considering the right action

$$R_{\exp(tA)}(u) = u \exp(tA), \tag{2.48}$$

which defines a curve through u in P. Since $\pi(u) = \pi(u \exp(tA)) = x$, this curve lies entirely within the fiber at point $x \in X$. The horizontal subspace $H_u P$ is the complement of $V_u P$ in $T_u P$. As stated by the following Definition, a connection is a unique splitting, point by point, in horizontal and vertical subspaces.

Definition 2.3.7 (Connection)**.** *Let $P(X, \mathrm{G})$ be a principal* G*-bundle and $(\mathrm{G}, \cdot)$ a Lie group. A connection on P is a unique separation of the tangent space $T_u P$ into the vertical subspace $V_u P$ and the horizontal subspace $H_u P$ such that:*

(i) $T_u P = H_u P \oplus V_u P, \qquad \forall\, u \in P$;

(ii) any smooth vector field W on P is separated into smooth vector fields $W_u^H \in H_u P$ and $W_u^V \in V_u P$ such that

$$W_u = W_u^H + W_u^V;$$

(iii) $H_{ug} P = (R_g)_* H_u P$ for arbitrary $u \in P$ and $g \in G$, where $R_g : P \times \mathrm{G} \to P$ denotes the right action of G *on P.*

We now introduce a useful construction to build up the connection 1-form which is a systematic way we implement the splitting above.

Definition 2.3.8 (Fundamental vector field on P)**.** *The fundamental vector on P at u is defined by*

$$\mathfrak{X}_u^{\#}(f(u)) := \frac{d}{dt} f\left(u \exp(t\mathfrak{X})\right)\bigg|_{t=0}$$

where $\mathfrak{X} \in \mathfrak{g}$. The vector $\mathfrak{X}_u^{\#}$ is tangent to P at u, hence $\mathfrak{X}_u^{\#} \in V_u P$, and in this way we define a vector field $\mathfrak{X}^{\#}$, called the fundamental vector field on P generated by $\mathfrak{X}$. There is a vector space isomorphism

$$\# : \mathfrak{g} \to V_u P, \quad \mathfrak{X} \mapsto \mathfrak{X}_u^{\#}.$$

We can now introduce the connection 1-form that is a $\mathfrak{g}$-valued one-form $\omega \in \mathfrak{g} \otimes \Gamma(T^* P)$.

Definition 2.3.9 (Connection 1-form)**.** *A connection 1-form $\omega \in \mathfrak{g} \otimes \Gamma(T^* P)$ is a projection of the tangent space $T_u P$ onto the vertical component $V_u P \simeq \mathfrak{g}$. The projection properties are summarized by the following*

1. $\omega(\mathfrak{X}_u^{\#}) = \mathfrak{X} \qquad \forall\ \mathfrak{X} \in \mathfrak{g},\ \forall\ u \in P$;
2. $(R_g)^*(\omega) = \mathrm{Ad}_{g^{-1}}(\omega)$.

The next Proposition shows that the notions of connection introduced in Definitions 2.3.7 and 2.3.9 are equivalent.

Proposition 2.3.1 (Link between connection 1-form and connection)**.** *The horizontal subspaces defined by*

$$H_uP := \{W_u \in T_uP \mid \omega(W_u) = 0\}$$

satisfy the condition

$$(R_g)_*H_uP = H_{ug}P.$$

Proof. Fix a point $u \in P$ and define H_uP as the kernel of the connection 1-form. Take $W_u \in H_uP$ and consider the pushforward $(R_g)_*(W_u) \in T_{ug}P$. We compute:

$$\omega((R_g)_*(W_u)) = ((R_g)^*(\omega))(W_u) = \mathrm{Ad}_{g^{-1}}(\omega(W_u)) = g^{-1}\omega(W_u)g = 0,$$

since $\omega(W_u) = 0$. Therefore, $(R_g)_*(W_u) \in H_{ug}P$, which shows that $(R_g)_*H_uP \subseteq H_{ug}P$.

Moreover, $(R_g)_*$ is an invertible linear map, so the inclusion is actually an equality. Hence, for any vector $Z_{ug} \in H_{ug}P$, there exists $W_u \in H_uP$ such that $Z_{ug} = (R_g)_*(W_u)$. □

The connection 1-form ω defined here is known as the Ehresmann connection in the literature [36].

We now want to give a local expression of the connection 1-form. This is achieved by defining first a local $\mathfrak{g}$-valued 1-form on X by taking the pullback using a section of the principal G-bundle and then using it to define the local version of the connection 1-form.

Let $\{U_i\}$ be an open covering of X, and let s_i be a local section defined on each U_i. We introduce a $\mathfrak{g}$-valued 1-form A_i on U_i defined by

$$A_i := (s_i)^*(\omega) \in \mathfrak{g} \otimes \Omega^1(U_i). \tag{2.49}$$

This is what in Physics is known as the gauge potential (Yang-Mills potential) in form notation: the index "i" indicates that this is the local form on the trivializing open U_i; we can expand it in the basis of differentials and get the gauge field components. The local version of the connection 1-form ω is defined by

$$\omega_i := g_i^{-1}\, \pi^*(A_i)\, g_i + g_i^{-1}\, d_P g_i \tag{2.50}$$

where d_P is the exterior derivative on P, and g_i is the canonical local trivialization defined by

$$\varphi_{U_i}^{-1}(u) = (x, g_i), \tag{2.51}$$

since, for $u \in \pi^{-1}(x)$ with $x \in U_i$, there exists a unique element $g_i \in \mathrm{G}$ such that $u = s_i(x)g_i$. The following Theorem ensures that given a $\mathfrak{g}$-valued 1-form A_i on U_i, we can reconstruct a connection 1-form ω whose pullback by s_i is exactly A_i.

Theorem 2.3.1 (Construction of the 1-form from the gauge potential)**.** *Given a $\mathfrak{g}$-valued 1-form ω_i on U_i and a local section $s_i : U_i \to \pi^{-1}(U_i)$, there exists a connection 1-form ω on P such that*

$$A_i = (s_i)^*(\omega).$$

For the connection 1-form ω to be defined uniquely on P, i.e. for the separation $T_uP = H_uP \oplus V_uP$ to be unique, we must have $\omega_i = \omega_j$ on $U_i \cap U_j$, in such a way that by the sheaf properties it glues to a single connection 1-form ω such that $\omega|_{U_i} = \omega_i$. To fulfil this condition, the local forms A_i must satisfy a specific transformation property, analogous to that of the Christoffel symbols:

$$A_j = g_{U_iU_j}^{-1}\, A_i\, g_{U_iU_j} + g_{U_iU_j}^{-1}\, dg_{U_iU_j}, \tag{2.52}$$

where $g_{U_iU_j}$ are the transition functions for the G-principal bundle. We note again that ω is defined globally over the bundle $P(X, \mathrm{G})$. Although there are many connection 1-forms on $P(X, \mathrm{G})$, they share the same global information about the bundle. In contrast, an individual local piece (the gauge potential) A_i is associated with the trivial subbundle $\pi^{-1}(U_i)$ and cannot have any global information on P. It is ω or, equivalently, the total of $\{A_i\}$ satisfying the compatibility condition (2.52), which carries the global information about the bundle. The crucial point is that since a non-trivial principal G-bundle does not admit a global section, the pullback $A_i = (s_i)^*\omega$ exists locally but not necessarily globally unless the principal bundle is trivial.

We now want to introduce the parallel transport which is a closely related notion to that of connection. We start with the following Definition.

Definition 2.3.10 (Horizontal lift)**.** *Let $P(X, \mathrm{G})$ be a principal* G*-bundle and let $\gamma : [0, 1] \to X$ be a curve in X. A curve $\tilde{\gamma} : [0, 1] \to P$ is said to be a horizontal lift of γ if*

$$\pi \circ \tilde{\gamma} = \gamma$$

and the tangent vector to $\tilde{\gamma}(t)$ always belongs to $H_{\tilde{\gamma}(t)}P$.

Let $\tilde{X}$ be a tangent vector to $\tilde{\gamma}$. Then it satisfies $\omega(\tilde{X}_{\tilde{\gamma}(0)}) = 0$ by definition. This condition is an ordinary differential equation (ODE) and the fundamental theorem of ODEs guarantees the local existence and uniqueness of the horizontal lift, as shown in Appendix C.

Thanks to the horizontal lift we can define the parallel transport.

Definition 2.3.11 (Parallel transport)**.** *Let $\gamma : [0, 1] \to X$ be a curve in X. Take a point $u_0 \in \pi^{-1}(\gamma(0))$; which specifies a unique horizontal lift $\tilde{\gamma}(t)$ of $\gamma(t)$ through u_0 then there exists unique point $u_1 = \tilde{\gamma}(1) \in \pi^{-1}(\gamma(1))$. The point u_1 is called the parallel transport of u_0 along the curve γ. This defines a map*

$$\Gamma(\tilde{\gamma}) : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1)), \quad u_0 \mapsto u_1$$

whose local form is

$$u_1 = s_i(\gamma(1))\mathcal{P}\left[\exp\left(-\int_0^1 (A_i)_\mu \frac{dx^\mu(\gamma(t))}{dt}\, dt\right)\right],$$

called parallel transport map.

In Appendix C are stated some important properties of the parallel transport. In the case of the associated $\mathbb{K}$-vector bundle to the principal $\mathrm{GL}(m, \mathbb{K})$-bundle

$P(X, \mathrm{GL}(m, \mathbb{K}))$, the parallel transport map of a vector $v \in V$ along the curve $\gamma : [0,1] \to X$, is expressed in local form as

$$v(1) = v(0) d\rho \left(\mathcal{P} \left[\exp \left(- \int_0^1 (A_i)_\mu(\gamma(t)) \, \frac{dx^\mu}{dt} \, dt \right) \right] \right) ,$$

where $\rho : \mathrm{G} \to \mathrm{GL}(V)$ is the representation defining the associated vector bundle. An important construction dealing with parallel transport is that of the holonomy group, which among other things, enters into the definition of Calabi-Yau manifolds as a non-local geometric structure. Take a point $u \in P$ with $\pi(u) = x$, and consider the set of loops $\mathcal{C}_x(X)$ at x defined by

$$\mathcal{C}_x(X) := \{\gamma : [0,1] \to X \mid \gamma(0) = \gamma(1) = x\}; \tag{2.53}$$

the set of elements

$$\mathrm{G}_u := \{g \in \mathrm{G} \mid \Gamma_{\mathrm{loop}(u)}(\tilde{\gamma}) = ug, \ \gamma \in \mathcal{C}_x(X)\}, \tag{2.54}$$

where $\Gamma_{\mathrm{loop}(u)}(\tilde{\gamma})$ is the parallel transport induced by $\gamma \in \mathcal{C}_x(X)$ such that $\tilde{\gamma}(0) = u$, is a subgroup of the structure group G of the principal bundle and it is called the holonomy group at u.
We can now give the following result that elucidates the importance of homotopic maps in the pullback construction and will be fundamental in the homotopy classification of complex vector bundle.

Theorem 2.3.2 (Isomorphism between pullback bundles via homotopic maps)**.** *Let (E, π, X) be a $\mathbb{K}$-vector bundle with fiber $\mathbb{K}^m$, and let f and g be homotopic maps from Y to X. Then the pullback bundles $\mathcal{F}^*\mathcal{E}$ and $\mathcal{G}^*\mathcal{E}$ are isomorphic $\mathbb{K}$-vector bundles over Y.*

Proof. Since f and g are homotopic, there exists

$$\Phi : Y \times [0,1] \to X$$

with $\Phi_0 = f$ and $\Phi_1 = g$. Let us consider the pullback bundle by Φ_t which is a bundle over $Y \times [0,1]$. This is a bundle such that at $t = 0$ we have $\mathcal{F}^*\mathcal{E}$ while at $t = 1$ we have $\mathcal{G}^*\mathcal{E}$. Let us take any connection, then the parallel transport for fixed x gives us the isomorphism between $\mathcal{F}^*\mathcal{E}$ and $\mathcal{G}^*\mathcal{E}$. □

In Appendix D we discuss curvature and Chern classes going deep into Chern-Weil theory.

2.3.3 Geometric phases

The notion of a phase acquired by a quantum system during evolution has long been recognized. However, for many decades it was thought that only the dynamical phase, arising from the time evolution governed by the Schrödinger equation, played a physical role. This view changed dramatically with the pioneering works of Pancharatnam and Berry [37, 38], who showed that quantum systems subject to adiabatic and cyclic evolution can acquire a phase of purely geometric and/or topological origin called Berry or geometric phase. This Berry phase triggered a wealth of developments across physics and mathematics, establishing deep connections with differential geometry, gauge theory and topology. It has found applications ranging from condensed matter physics to quantum computation, and from optics to atomic and molecular systems. The general concept of geometric phase was developed in adiabatically deformed quantum dynamics[3]. Consider a quantum system with Hamiltonian H_k that depends smoothly on a set of parameters $k \in \mathcal{M}$, where $\mathcal{M}$ is a parameter manifold. Let $|u(k)\rangle$ be a non-degenerate instantaneous eigenstate of H_k. If the parameters evolve adiabatically along a closed path $\mathcal{C} \subset \mathcal{M}$, the system returns to its initial physical state up to a phase which consists of two parts: one representing the temporal evolution and one not related to the temporal evolution which represents the geometric phase $\gamma_n[\mathcal{C}]$ associated with the path. This geometric phase depends only on the geometry of the path in parameter space and not on how fast it is traversed (as long as the adiabatic condition holds). It can be viewed as a manifestation of holonomy in a line bundle over $\mathcal{M}$ associated to a specific connection on this line bundle: Berry connection. The Berry connection associated to n-th fibre is defined as a 1-form on $\mathcal{M}$ by

$$\mathcal{A}_n := i\langle u_n(k), \partial_k u_n(k)\rangle, \tag{2.55}$$

which plays the role of a gauge potential. The corresponding curvature 2-form is known as the Berry curvature and it is analogous to the field strength in gauge theory. Therefore the geometric phase is also known as Berry phase. In other words, adiabatic theorem ensures that a quantum system remains within the same instantaneous eigenspace when the parameters vary slowly enough. This physical behavior naturally induces a rule of parallel transport on the eigenstates as the parameters change along a path in parameter space and, mathematically, this defines a connection on the complex line bundle whose fibers are spanned by the instantaneous eigenstates. This connection, i.e. the Berry connection, encodes how the phase of the eigenstate must be adjusted under infinitesimal changes in the parameters in order to preserve parallel transport. When the parameters trace a closed loop $\mathcal{C}$ in parameter space, the state returns to its initial eigenspace but may acquire an additional geometric phase, i.e. the Berry phase. This phase is given by the holonomy of the Berry connection along the loop

$$\gamma_n[\mathcal{C}] = \oint_{\mathcal{C}} \mathcal{A}_n\,. \tag{2.56}$$

Thus, the Berry phase represents the intrinsic geometric information contained in the adiabatic evolution: it measures how the local definition of the phase of the eigenstate fails to be globally consistent when transported around a closed path.

For our purposes, we are interested in Berry connection of the Bloch bundle where $\mathcal{M} = \mathbb{T}^d$ and we are going to consider the total Berry connection

$$\mathcal{A} := i\sum_n \langle u_n(k), \partial_k u_n(k)\rangle, \tag{2.57}$$

in order to taken into account the total Berry phase.

[3]The adiabatic theorem is a concept in quantum mechanics which states that a physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum [39].

Chapter 3

Topological quantum matter

In recent decades, a new paradigm in condensed matter physics has emerged with the discovery of topological phases of matter. Unlike conventional phases characterized by local order parameters and spontaneous symmetry breaking, topological phases are defined by global, topological invariants that remain robust under continuous deformations of the system. These phases exhibit quantized physical responses and host exotic surface states protected by topological features of the bulk system. The study of topological matter has deepened the connection between quantum physics and abstract mathematical frameworks, especially topology and K-theory, leading to both theoretical advances and technological implications. The origins of topological condensed matter can be traced back to the 1980s, with the discovery of the integer and fractional quantum Hall effects[1]. These phenomena defied conventional band theory and were soon understood to be topologically non-trivial thanks to the introduction of topological invariants such as the first Chern number. In the 2000s, the theoretical prediction and subsequent experimental realization of topological insulators brought topological phases into the spotlight once again. Topological insulators are materials that behave as insulators in their bulk but possess conducting states on their boundaries, which are protected by symmetries and topological invariants. Examples of 3D topological insulators include Bi_2Se_3, Bi_2Te_3, and Sb_2Te_3. These materials exhibit spin-momentum locked surface states that are robust against non-magnetic disorder. Applications of topological insulators are being explored in fields such as spintronics, low-power electronics and quantum computing, particularly in the engineering of fault-tolerant qubits using Majorana modes in topological materials. The unique transport properties and robustness to perturbations make these materials candidates for novel technological platforms. Future prospects include the integration of topological insulators in hybrid devices, the exploration of higher-order topological phases, and the manipulation of topological features using external parameters such as pressure or electromagnetic fields. To achieve a systematic understanding and control of topological phases a rigorous classification is essential. Mathematically, non-interacting topological insulators can be modeled

[1]The integer quantum Hall effect occurs in 2D electron systems at low temperatures and under strong magnetic fields, exhibiting a quantized transverse conductance in integer multiples of e^2/h. The fractional quantum Hall effect displays similar quantization in rational fractions of e^2/h, arising from strong electron-electron interactions that give rise to highly correlated quantum states with fractional charge excitations.

using Bloch bundles. The classification of such bundles up to isomorphism or stable isomorphism naturally incorporates the use of K-theory, which allows one to assign well-defined invariants to these phases. The problem becomes more intricate in the presence of discrete symmetries like time reversal, particle hole, or chiral symmetry, which modify the structure of the bundle and thus influence the classification.

In Section 3.1 we introduce the fundamentals of K-theory. After a brief motivation and connection with generalized cohomology theories, we define the Grothendieck group associated to a topological space and construct the reduced and unreduced K^0-groups. We distinguish between bundle isomorphisms and stable isomorphisms, demonstrating that the latter correspond to elements of the reduced K^0-group. We also introduce higher K-groups and explain the Bott periodicity theorem, motivating why K-theory constitutes a generalized cohomology theory. This can be considered as a confirmation of the general considerations in bundle theory from the previous Chapter.

In Section 3.2 we examine how discrete symmetries alter the construction of Bloch bundles. We introduce the most relevant, for us, discrete symmetry classes: time reversal symmetry, particle hole symmetry and chiral symmetry. These symmetries impose constraints on the Hamiltonian and on the spectral eigenprojectors modifying the classification problem in profound ways.

In Section 3.3 we focus on symmetric Bloch bundles and we give the definition of topological insulators. We discuss the "ten-fold way", a comprehensive classification scheme that categorizes topological insulators based on symmetry classes and spatial dimensions. This scheme relies on twisted equivariant K-theory, which captures both the topological and symmetry properties of the system.

The main references for this Chapter are [5, 6, 32, 40, 41, 42, 43, 44].

3.1 Introduction to K-theory

In algebraic topology, classical cohomology assigns to each topological space X a family of abelian groups $H^\bullet(X)$ that capture topological invariants such as, for example, holes. A generalized cohomology theory does the same, but uses more sophisticated tools to define groups $E^\bullet(X)$ more general than classical $H^\bullet(X)$. To introduce correctly the idea of a generalized cohomology theory we need to discuss the Eilenberg-Steenrod axioms.

Definition 3.1.1 (Eilenberg-Steenrod axioms)**.** *A classical cohomology theory is a sequence of functors H^n from the category of pairs (X, A) of topological spaces to the category of abelian groups $H^n(X, A)$ that must satisfy the following axioms*

1. *Functoriality axiom. Each continuous map $f : (X, A) \to (Y, B)$ induces a map in cohomology*
$$f^* : H^n(Y, B) \to H^n(X, A).$$

2. *Homotopy invariance axiom: If f and g are two continuous homotopical functions, then $f^* = g^*$.*

3. *Long exact sequence axiom. For each triple (X, A, B) with $B \subseteq A \subseteq X$ there is a long exact sequence*

$$\cdots \to H^n(A, B) \to H^n(X, B) \to H^n(X, A) \to H^{n+1}(A, B) \to \cdots$$

4. *Disjoint union axiom. Given $X = \bigsqcup_\alpha X_\alpha$ then*

$$H^n(X) \simeq \bigoplus_\alpha H^n(X_\alpha).$$

5. *Dimension axiom.*

$$H^n(\mathrm{pt}) = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \neq 0. \end{cases}$$

A generalized cohomology theory is a sequence of functors E^n that assigns to each topological space X a family of abelian groups $E^n(X)$ for every integer n, satisfying all of the Eilenberg-Steenrod axioms except the Dimension axiom. K-theory is an example of a generalized cohomology theory that studies quaternionic, complex or real vector bundles over a topological space X. Moreover, generalized cohomology theories find applications in Physics not only in Condensed Matter Physics as we are going to discuss in a while but also, for example, in the classification of D_p-brane charges in String Theory and more generally in the classification of topological charges associated to classical and quantum fields.

In this section our main scope is to define the reduced K^0-group. However, we first need to recall the definition of Grothendieck group of a semigroup.

Definition 3.1.2 (Grothendieck group)**.** *If $(S, +)$ is a (commutative) semigroup, the Grothendieck group $(G(S), +)$ is defined as the set*

$$G(S) := S \times S / \sim_G,$$

where $(a, b) \sim_G (a', b') \in S \times S$ if there exists $c \in S$ such that

$$a + b' + c = a' + b + c.$$

The equivalence class of a pair $(a, b) \in S \times S$ in $G(S)$ is denoted by $a - b$. Each element $a \in S$ is identified with the element $a - 0 = a \in G(S)$, where 0 is the neutral element of $(S, +)$. The group structure on $G(S)$ is defined by

$$(a - b) + (c - d) := (a + c) - (b + d),$$

hence the sum of equivalence classes is the equivalence class of the sums.

The basic example of a Grothendieck group is the group $(\mathbb{Z}, +)$ built from the semigroup $(\mathbb{N}, +)$. We now want to use the construction of Definition 3.1.2 to the following semigroup

$$\mathrm{Vect}(X) := \bigcup_{m \in \mathbb{N}} \mathrm{Vect}_m(X),$$

where $\mathrm{Vect}_m(X)$ is the set of isomorphism classes of $\mathbb{K}$-vector bundles of rank m over X where $\mathbb{K}$ is the real or complex field or the quaternionic sfield[2]. In the case of a condensed matter system the space X is the Brillouin torus associated to the system. This set carries the structure of a semigroup under the Whitney sum operation $\oplus$, which means that we can define the sum $\mathcal{E} \oplus \mathcal{F}$ of $\mathbb{K}$-vector bundles looking at Definition 2.3.4. Moreover, this operation has a neutral element if we consider the $\mathbb{K}$-vector bundle of rank 0; therefore $\mathrm{Vect}(X)$ the algebraic structure of a monoid. However, contrary to what happens in a group, this operation does not have inverses. Therefore we consider the Grothendieck group $G(\mathrm{Vect}(X))$ which is abelian since the Whitney sum is commutative.

Definition 3.1.3 (K^0-group of X)**.** *The K^0-group of X is*

$$K^0(X) := G(\mathrm{Vect}(X)).$$

Elements of the K^0-group of X are of the form $\mathcal{E} \ominus \mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are $\mathbb{K}$-vector bundles on X and where $(\mathcal{E}, \mathcal{F}) \sim_G (\mathcal{E}', \mathcal{F}')$ if $\mathcal{E} \oplus \mathcal{F}' \oplus \mathcal{G} \simeq \mathcal{E}' \oplus \mathcal{F} \oplus \mathcal{G}$ for some $\mathbb{K}$-vector bundle $\mathcal{G}$ on X.

Proposition 3.1.1. *The K^0-group of X is such that*

$$K^0(X) \simeq \mathrm{Ker}(\mathrm{Rk} : K^0(X) \to \mathbb{Z}) \oplus \mathbb{Z}.$$

Proof. The rank map is such that

$$\mathrm{rk}(\mathcal{E} \oplus \mathcal{F}) = \mathrm{rk}(\mathcal{E}) + \mathrm{rk}(\mathcal{F}),$$

hence the map

$$\mathrm{rk} : \mathrm{Vect}(X) \to \mathbb{N}$$

is a semigroup homomorphism. Since both the semigroups $\mathrm{Vect}(X)$ and $\mathbb{N}$ can be extended to a group by the Grothendieck construction, the rank map induces a group homomorphism between the Grothendieck groups of the semigroups

$$\mathrm{Rk} : K^0(X) \to \mathbb{Z}, \quad \mathcal{E} \ominus \mathcal{F} \mapsto \mathrm{rk}(\mathcal{E}) - \mathrm{rk}(\mathcal{F}).$$

Moreover, every $\mathbb{K}$-vector bundle $\mathcal{E}$ is identified with the class $\mathcal{E} \ominus 0$ in $K^0(X)$, with 0 being the trivial bundle of null rank $\mathcal{E}_{\mathrm{tri}}^{(0)}$. In particular, we can define a group homomorphism

$$\varepsilon : \mathbb{Z} \to K^0(X), \quad m \mapsto \mathcal{E}_{\mathrm{tri}}^{(m)}.$$

To prove this is a group homomorphism we need to verify that ε respects the sum, that is, for every $m_1, m_2 \in \mathbb{Z}$, the following equality holds:

$$\varepsilon(m_1 + m_2) = \varepsilon(m_1) \oplus \varepsilon(m_2).$$

By definition

[2]In the case of $\mathbb{K} = \mathbb{H}$ the quaternionic sfield, i.e. division ring, the fibers are not vector spaces but instead modules. In the following the we will use the abuse of notation "$\mathbb{K}$-vector bundle" also to indicate the $\mathbb{H}$-module bundle unless otherwise specified.

$$\varepsilon(m_1+m_2) = \mathcal{E}_{\text{tri}}^{(m_1+m_2)}.$$

while

$$\varepsilon(m_1) := \mathcal{E}_{\text{tri}}^{(m_1)}, \qquad \varepsilon(m_2) := \mathcal{E}_{\text{tri}}^{(m_2)}.$$

The sum in $K^0(X)$ is the direct sum of vector bundles, so

$$\varepsilon(m_1) \oplus \varepsilon(m_2) = \mathcal{E}_{\text{tri}}^{(m_1)} \oplus \mathcal{E}_{\text{tri}}^{(m_2)} = \mathcal{E}_{\text{tri}}^{(m_1+m_2)}.$$

Moreover, $\mathrm{Rk} \circ \varepsilon = 1_{\mathbb{Z}}$ since

$$\mathrm{Rk}(\varepsilon(m)) = \mathrm{Rk}(\mathcal{E}_{\text{tri}}^{(m)}) = \mathrm{Rk}(\mathcal{E}_{\text{tri}}^{(m)} \ominus 0) = m;$$

therefore, considering the following short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\mathrm{Rk} : K^0(X) \to \mathbb{Z}) \xrightarrow{\iota} K^0(X) \xrightarrow{\mathrm{Rk}} \mathbb{Z} \longrightarrow 0,$$

the map ε provide a right split and by the splitting lemma the sequence split. □

The above proof motivates the following Definition.

Definition 3.1.4 (Reduced K^0-group)**.** *The reduced K^0-group of X is*

$$\tilde{K}^0(X) := \mathrm{Ker}(\mathrm{Rk} : K^0(X) \to \mathbb{Z}) \subset K^0(X).$$

The reduced K^0-group is the part of the K^0-group at fixed rank, which means that its elements are such that they have all the same rank since it is defined as the kernel of the rank map.
We now introduce an equivalence relation, weaker than the isomorphism between $\mathbb{K}$-vector bundles, that is crucial in the K-theory classification of $\mathbb{K}$-vector bundles. This relation is called stable equivalence: given two $\mathbb{K}$-vector bundles $\mathcal{E}$ and $\mathcal{F}$ over X we have $\mathcal{E} \sim_{\text{seq}} \mathcal{F}$ if $\mathcal{E} \oplus \mathcal{E}_{\text{tri}}^{(m)} \simeq \mathcal{F} \oplus \mathcal{E}_{\text{tri}}^{(n)}$. The following Theorem demystifies the elements of the group $\tilde{K}^0(X)$.

Theorem 3.1.1. *There exists an isomorphism*

$$\mathrm{Vect}(X)/{\sim_{\text{seq}}} \simeq \tilde{K}^0(X).$$

Proof. We define the map

$$\alpha : \mathrm{Vect}(X) \to \tilde{K}^0(X), \quad \mathcal{E} \mapsto \mathcal{E} \ominus \varepsilon(\mathrm{Rk}(\mathcal{E})).$$

We are going to show that this map induces the desired isomorphism.
By definition, a class $\mathcal{E} \ominus \mathcal{F} \in K^0(X)$ is in $\tilde{K}^0(X)$ if and only if $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{F})$. Let then $\mathcal{E} \ominus \mathcal{F} \in \tilde{K}^0(X)$ and, thanks to Corollary E.0.1, let $\mathcal{F}'$ be a $\mathbb{K}$-vector bundle over X such that $\mathcal{F} \oplus \mathcal{F}' \simeq \mathcal{E}_{\text{tri}}^{(N)}$. By definition of the Grothendieck group

$$\mathcal{E} \ominus \mathcal{F} = (\mathcal{E} \ominus \mathcal{F}) \oplus (\mathcal{F}' \ominus \mathcal{F}') = (\mathcal{E} \oplus \mathcal{F}') \ominus (\mathcal{F} \oplus \mathcal{F}') = (\mathcal{E} \oplus \mathcal{F}') \ominus \varepsilon(\mathrm{rk}(\mathcal{E} \oplus \mathcal{F}')) = \alpha(\mathcal{E} \oplus \mathcal{F}'),$$

since $(\mathcal{F}' \ominus \mathcal{F}')$ is the trivial class. Hence α is surjective since every class $\mathcal{E} \ominus \mathcal{F} \in \tilde{K}^0(X)$ is in the image of α. Now, we have to show that if $\alpha(\mathcal{E}) = \alpha(\mathcal{F}) \in \tilde{K}^0(X)$, then $\mathcal{E} \sim_{\text{seq}} \mathcal{F}$. Let $m := \text{Rk}(\mathcal{E})$ and $n := \text{Rk}(\mathcal{F})$, so that

$$\varepsilon(\text{Rk}(\mathcal{E})) = \mathcal{E}_{\text{tri}}^{(m)}, \quad \varepsilon(\text{Rk}(\mathcal{F})) = \mathcal{E}_{\text{tri}}^{(n)}.$$

Then $\mathcal{E} \ominus \mathcal{E}_{\text{tri}}^{(m)} = \mathcal{F} \ominus \mathcal{E}_{\text{tri}}^{(n)}$ in the Grothendieck group if there exists a $\mathbb{K}$-vector bundle $\mathcal{G}$ over X such that

$$\mathcal{E} \oplus \mathcal{E}_{\text{tri}}^{(n)} \oplus \mathcal{G} \simeq \mathcal{F} \oplus \mathcal{E}_{\text{tri}}^{(m)} \oplus \mathcal{G}.$$

Let now $\mathcal{G}'$ be a $\mathbb{K}$-vector bundle over X such that $\mathcal{G} \oplus \mathcal{G}' \simeq \mathcal{E}_{\text{tri}}^{(N)}$. It follows that

$$\mathcal{E} \oplus \mathcal{E}_{\text{tri}}^{(n+N)} \simeq \mathcal{E} \oplus \mathcal{E}_{\text{tri}}^{(n)} \oplus \mathcal{G} \oplus \mathcal{G}' \simeq \mathcal{F} \oplus \mathcal{E}_{\text{tri}}^{(m)} \oplus \mathcal{G} \oplus \mathcal{G}' \simeq \mathcal{F} \oplus \mathcal{E}_{\text{tri}}^{(m+N)}$$

Hence $\mathcal{E}$ is stably equivalent to $\mathcal{F}$. □

Therefore, the elements in the reduced K^0-group can be regarded as stable equivalence classes of $\mathbb{K}$-vector bundles over X, i.e. $\mathbb{K}$-vector bundles which become isomorphic after taking a direct sum with trivial $\mathbb{K}$-vector bundles. Isomorphic $\mathbb{K}$-vector bundles are clearly also stably equivalent; however, the converse does not hold in general: the stable equivalence classes of $\mathbb{K}$-vector bundles are strictly larger than their isomorphism classes.
A standard example of a real vector bundle not isomorphic to the trivial vector bundle but that is stably equivalent to the trivial real bundle is given by the tangent bundle to the 2-sphere, $\mathcal{T}S^2$. The non-triviality of this bundle follows from the hairy ball theorem, which states that every smooth vector field on the sphere must vanish somewhere hence showing the non-existence of a global frame for $\mathcal{T}S^2$. However, the normal bundle $\mathcal{N}S^2$ is isomorphic to the trivial real vector bundle, since the 2-sphere is orientable. Therefore, we obtain

$$\mathcal{T}S^2 \oplus \mathcal{E}_{\text{tri}}^{(1)} \simeq \mathcal{T}S^2 \oplus \mathcal{N}S^2 \simeq \mathcal{T}\mathbb{R}^3|_{S^2} \simeq \mathcal{E}_{\text{tri}}^{(3)} \simeq \mathcal{E}_{\text{tri}}^{(2)} \oplus \mathcal{E}_{\text{tri}}^{(1)}.$$

There are, however, situations in which stable equivalence and isomorphism coincide: this occurs when the rank of the $\mathbb{K}$-vector bundle is sufficiently high relative to the dimension of the base space, the so-called stable range. In particular, for complex vector bundles the notion of stable equivalence reduces to that of isomorphism if the dimension of the base space is at most 2.
We can also define the higher K-groups

Definition 3.1.5 (Higher K-groups)**.** *The higher K-groups are defined as*

$$K^n(X) := K^0(\text{Susp}_n(X)), \qquad \forall\ n \in \mathbb{N}^+$$

where $\text{Susp}_1(X) := X \times [0,1]/((X \times \{0\}) \cup (X \times \{1\})$ *is called the suspension of* X *and* $\text{Susp}_n(X) := \text{Susp}_1(\text{Susp}_{n-1}(X))$.

The suspension of a topological space X, $\text{Susp}_1(X)$, can be constructed as follows: construct the cylinder of X, given by $X \times [0,1]$, then identify $X \times \{0\}$ and $X \times \{1\}$ to single points. A fundamental result in K-theory, due to Bott theorems that describe a periodicity in the homotopy groups of classical groups, states that there is a periodicity in the K-theory groups.

Theorem 3.1.2 (Bott). *There exists periodicity in the K-theory groups*

1. *in the K-theory of complex vector bundles there is a 2-periodicity*

$$K^2(X) \simeq K^0(X), \qquad \tilde{K}^2(X) \simeq \tilde{K}^0(X);$$

2. *in the K-theory of real vector bundles there is a 8-periodicity*

$$K^8(X) \simeq K^0(X), \qquad \tilde{K}^8(X) \simeq \tilde{K}^0(X);$$

3. *in the K-theory of quaternionic module bundles there is a 8-periodicity*

$$K^8(X) \simeq K^0(X), \qquad \tilde{K}^8(X) \simeq \tilde{K}^0(X).$$

Bott periodicity explains why K-theory is a generalized and not a classical cohomology theory. Indeed, in K-theory we have

$$K^0(\mathrm{pt}) \simeq \mathbb{Z},$$

since $\mathbb{K}$-vector bundles over a point are just $\mathbb{K}$-vector spaces and their isomorphism classes are determined by their dimension, i.e.

$$\mathrm{Vect}(\mathrm{pt}) \simeq \mathbb{N};$$

thus, passing to the Grothendieck group, $K^0(\mathrm{pt}) \simeq \mathbb{Z}$. Since there is a periodicity we must have a non-trivial higher K-group for pt and the Dimension axiom is violated.

3.2 Time reversal, charge conjugation and chiral symmetries

One of the crucial features of topological matter is that quantum symmetries may impose certain non-trivial constraints on the family of spectral eigenprojections, which in turn can affect the topology of the corresponding Bloch bundle. As we saw, topological quantum systems display some form of translation invariance but, on top of that, other discrete symmetries may be imposed. The point is that the family of spectral eigenprojection $\{P_k\}_{k\in\mathbb{R}^d}$ defined in (2.30), which arises for the Zak-Bloch-Floquet representation of periodic quantum systems (for example in the case a lattice in the one-body approximation) and is associated with the Bloch bundle in Definition 2.2.6, must satisfy further conditions when the quantum system under consideration shows further symmetries such as time reversal, charge conjugation and/or chiral symmetry.

Time reversal symmetry

A time reversal symmetry is a projective anti-unitary representation on the Hilbert space of the quantum system $\mathcal{H}$ of the group $(\mathbb{Z}_2 = \{\pm 1\}, \cdot)$, specified by the action of an operator T that represent the non-trivial element $-1 \in \mathbb{Z}_2$. This operator T, which is in fact U_{-1} with the notations on the first Chapter, is called the time reversal operator. It is required to be a dynamical symmetry of the system, that is, it commutes with the Hamiltonian H, since time reversal symmetry can be read off directly from the Schrödinger equation. Being a projective anti-unitary representation, we have that

$$T^2 = e^{i\vartheta} 1_{\mathcal{H}}, \qquad e^{i\vartheta} := \omega(-1,-1), \tag{3.1}$$

since $U_1 = 1_{\mathcal{H}}$ and $U^{-1}_{-1\cdot -1} U_{-1} U_{-1} = 1_{\mathcal{H}} T^2 = T^2$. By associativity and the fact that T is anti-unitary, we conclude that

$$\begin{aligned} T^3 &= T^2 T = e^{i\vartheta} T, \\ T^3 &= T T^2 = T e^{i\vartheta} = e^{-i\vartheta} T, \end{aligned} \tag{3.2}$$

that is,

$$e^{i\vartheta} = e^{-i\vartheta}, \tag{3.3}$$

which implies that $e^{i\vartheta} \in \{\pm 1\}$. A time reversal operator such that $T^2 = +1_{\mathcal{H}}$ is called even while one such that $T^2 = -1_{\mathcal{H}}$ is called odd.

Particle hole or Charge conjugation symmetry

In Quantum Mechanics, so in particular in Condensed Matter Physics, Dirac's theory implies a pairing between particles and holes (i.e. conducting and valence electrons in a solid). The definition of what is a "particle" and what is a "hole" is in fact arbitrary, hence the theory is required to be invariant under their exchange. This charge conjugation symmetry, or particle hole symmetry, is described as a projective anti-unitary representation of $(\mathbb{Z}_2, \cdot)$, specified by a charge conjugation operator C representing $-1 \in \mathbb{Z}_2$. As with a time reversal operator, the operator C is nothing but U_{-1} and two types of representations are possible depending on whether

$$C^2 = +1_{\mathcal{H}} \quad \text{or} \quad C^2 = -1_{\mathcal{H}}. \tag{3.4}$$

However, unlike time reversal, charge conjugation is not a dynamical symmetry in a strict sense: the charge conjugation operator anticommutes with the Hamiltonian

$$C^{-1} H C = -H. \tag{3.5}$$

This reflects the fact that particle states have opposite energy to their corresponding hole partners.

Chiral symmetry

The composition of a time reversal symmetry and a charge conjugation symmetry, which we could impose to commute with each other, yields a projective unitary representation of $(\mathbb{Z}_2, \cdot)$, characterized by the chiral operator

$$S := TC, \tag{3.6}$$

which also represents $-1 \in \mathbb{Z}_2$. Anyway, chiral symmetry need not necessarily arise from the combination of time reversal and charge conjugation symmetries since it can exists independently even if the other two anti-unitary symmetries are broken. Generally speaking, $S^2 = \pm 1_{\mathcal{H}}$, but by redefining $\tilde{S} := iS = e^{i\pi/2}S$, a chiral symmetry can always be seen as a representation in which

$$S^2 = +1_{\mathcal{H}}. \tag{3.7}$$

In the case in which $S := TC$, then $S^2 = (TC)^2(TC)^2$ with $T^2 = \epsilon_T 1_{\mathcal{H}}$ and $C^2 = \epsilon_C 1_{\mathcal{H}}$ where $\epsilon_T, \epsilon_C \in \{\pm 1\}$. Moreover, the chiral operator anticommutes with the Hamiltonian,

$$S^{-1}HS = -H,$$

so chiral symmetry is not a dynamical symmetry.
We can play with the commutativity/anti-commutativity of T and C to force $S^2 = +1_{\mathcal{H}}$.

3.3 Topological insulator and the ten-fold way

We now enter in the crucial stages in which we define our object of study: topological insulators.

Definition 3.3.1 (Topological insulator). *A topological insulator is a family $\{H_k\}_{k\in X}$ of self-adjoint operators acting on a Hilbert space $\mathcal{H}$, parametrized by a smooth manifold X, such that:*

1. *the operators H_k may be unbounded on $\mathcal{H}$, but they are uniformly bounded from below;*
2. *the set*
$$R := \{(k, z) \in X \times \mathbb{C} : z \in \rho(H_k)\}$$
is open, and the resolvent map
$$\rho : R \to \mathcal{B}(\mathcal{H}), \quad k, z \mapsto (H_k - z1_{\mathcal{H}_k})^{-1}$$
is smooth in k and analytic in z, taking values in the subalgebra of compact operators;
3. *there exist $\mu \in \mathbb{R}$ and $g > 0$ such that*
$$\inf_{k\in X} \inf \sigma(H_k) < \mu - \frac{g}{2}, \quad \text{and} \quad \left(\mu - \frac{g}{2}, \mu + \frac{g}{2}\right) \cap \sigma(H_k) = \emptyset \qquad \forall\ k \in X.$$
The interval $(\mu - \frac{g}{2}, \mu + \frac{g}{2}) \subset \mathbb{R}$ is called the spectral gap.

The first condition is essentially a physical one: the energy of the system cannot be unbounded from below since otherwise the condensed matter system will be unstable and it would be more convenient for the system to disintegrate into its elementary constituents as well as providing us with an infinite reserve of energy[3]. This condition can be relaxed to consider Dirac-like models in which the infinity of

[3]It is obviously absurd.

negative energy states is interpreted as the Dirac's sea. This interpretation finds its true fulfillment in Quantum Field Theory in which infinite negative energies are the prerogative of renormalization theory [45, 15]. The third condition is also a physical one since it tells us that the spectrum of the Hamiltonian is not a continuum but is interspersed with spectral gaps which, for example, separate the valence band from the conduction band. The name of "insulator" is due to the above particular structure, i.e. the presence of an open spectral gap separating valence and conductance energy bands. Without loss of generality, we can always shift the energy by a constant, so that the spectral gap occurs around $\mu = 0$. The "topological" aspects follow instead from the Bloch bundle associated to the range of the spectral eigenprojectors.
In the following table we give some examples of topological insulating materials with their respective chemical formulae.

Table 3.1. Examples of topological insulators with relative chemical formulae.

Material	**Chemical Formula**	**Dimension**
Bismuth selenide	Bi_2Se_3	3D
Bismuth telluride	Bi_2Te_3	3D
Antimony telluride	Sb_2Te_3	3D
Bismuth–antimony alloy	$Bi_{1-x}Sb_x$	3D
Mercury telluride / Cadmium telluride	HgTe/CdTe	2D
Stanene	Sn	2D

Several classification schemes have been proposed to distinguish the topological phases of quantum matter. The earlier approach states that two topological insulators are in the same topological phase if they can be continuously[4] deformed into one another without closing the spectral gap. Mathematically speaking, this corresponds to the existence of a continuous[5] homotopy between the respective families of spectral eigenprojections that, according to the homotopy classification Theorem E.0.3, implies that the associated Bloch bundles are isomorphic. However, such a classification is feasible only in some low dimensional cases and a K-theoretic approach gained prominence, in which only the K^0-classes of the Bloch bundle (or equivalently, its stable equivalence classes) are considered relevant for classification. From a physical standpoint, this is justified by the argument that the addition of topologically trivial energy bands should not affect the physical properties of the low lying bands below the gap. This viewpoint led to the development of the so-called periodic tables of topological insulators, which are essentially tables of the groups $K^0(X)$ for some topological spaces X, with the dimension of X playing a central role.

Without imposing any further symmetries on the system other than translational symmetry, the classification of topological insulators via K-theory is quite simple and it is summarized in the following table.

[4] In the norm-resolvent topology since we are talking about the Hamiltonians of the systems.
[5] In the standard norm topology since we are now talking about the spectral eigenprojections.

Table 3.2. Classification of free (full translation invariance) and translation-invariant (lattice translation invariance) fermion ground states in symmetry class A. In the first case there are no topological phases in odd d and integer classification in even d. In the second case there is a richer structure in higher dimensions.

Symmetry Setting	**Momentum Space**	**Topological Classification**
$\mathrm{G} = \mathbb{R}^d$	$X = S^d$	$\tilde{K}^0(S^d) = \begin{cases} 0 & \text{if } d \text{ odd} \\ \mathbb{Z} & \text{if } d \text{ even} \end{cases}$
$\mathrm{G} = \Gamma \cong \mathbb{Z}^d$	$X = \mathbb{T}^d$	$\tilde{K}^0(\mathbb{T}^d) = \begin{cases} 0 & \text{if } d = 1 \\ \mathbb{Z}^{2^d - 1} & \text{if } d > 1 \end{cases}$

As said before, quantum symmetries have a crucial role in enriching or trivializing the topology of the Bloch bundle. The type of discrete symmetries that are most relevant for topological insulators are time reversal symmetry T, charge conjugation symmetry C and chiral symmetry S. Whenever a system exhibits more than one of the above symmetries on top of translational symmetry, we require the corresponding symmetry operators to commute/anti-commute depending on the sign of the squared symmetry operator. Therefore, topological insulators can be categorized into distinct symmetry classes, each determined by the presence or absence of certain discrete symmetries. Taking into account all possible combinations of these symmetry properties leads to a total of ten distinct symmetry classes. This classification scheme is known as the "ten-fold way", and forms the foundation for understanding topological insulators in Condensed Matter Physics. From the relations between the T, S and C operators and the Hamiltonian, using the Zak-Bloch-Floquet representation we can derive how their fiberwise counterparts act on the fibered operators and thanks to the Riesz formula on the spectral eigenprojectors.

Proposition 3.3.1 (Discrete symmetries and fiber operators)**.** *Let T, C, S be, respectively, the time reversal, particle hole and chiral symmetry operator acting on the Hilbert space $\mathcal{H}$ and let us assume that they commute with every position operators X_i. Let $\mathfrak{T}, \mathfrak{C}, \mathfrak{S}$ be, respectively, their k-independent fiber operators with respect to the Zak-Bloch-Floquet transform that act on the fiber Hilbert space $\mathcal{H}_k$. Let H_k the fiber Hamiltonian and let P_k be the fiber of the spectral eigenprojector defined by the Riesz formula that projects onto all the negative energy bands. Then*

1. $H_{-k}\mathfrak{T} = \mathfrak{T}H_k, \quad \text{and} \quad P_k\mathfrak{T} = \mathfrak{T}P_{-k};$
2. $H_{-k}\mathfrak{C} = -\mathfrak{C}H_k, \quad \text{and} \quad P_k\mathfrak{C} = \mathfrak{C}(1_{\mathcal{H}_k} - P_{-k});$
3. $H_k\mathfrak{S} = -\mathfrak{S}H_k, \quad \text{and} \quad P_k\mathfrak{S} = \mathfrak{S}(1_{\mathcal{H}_k} - P_k).$

Proof. The relations in their first form are inherited by the fiber operators directly from the commutativity or anti-commutativity properties of the T, C, S operators with the Hamiltonian. Similarly, concerning the relations with the fiber operators of the spectral eigenprojectors, these follow directly from the former using the Riesz formula. □

The above intertwining relations for the spectral eigenprojections induce further structures on the associated Bloch bundle. This leads to the following general Definition.

Definition 3.3.2 (Symmetric Bundles). *Let $\mathbb{K}$ be a field and let (X, θ) be a smooth manifold equipped with an involution $\theta : X \to X$. We define*

1. *Time reversal symmetric bundle. A $\mathbb{K}$-vector bundle $\mathcal{E} = (E, \pi, X)$ over X is called time reversal symmetric if it is endowed with a map $\Theta : \mathcal{E} \to \mathcal{E}$ such that:*
   - *the following diagram commutes*

$$\begin{tikzcd} E \arrow[r, "\Theta"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "\theta"] & X \end{tikzcd}$$

   - *for every $k \in X$, the fiberwise map $\Theta_k : \mathcal{E}_k \to \mathcal{E}_{\theta(k)}$ is anti-unitary, and satisfies $\Theta_k^* \circ \Theta_k = \pm 1_{\mathcal{E}_k}$.*
2. *Charge conjugation symmetric bundle. A pair of $\mathbb{K}$-vector bundles $\mathcal{E}^- = (E^-, \pi^-, X)$ and $\mathcal{E}^+ = (E^+, \pi^+, X)$ over X are said to be charge conjugation symmetric if there exists a map $\Pi : \mathcal{E}^- \to \mathcal{E}^+$ such that:*
   - *the direct sum $\mathcal{E}^- \oplus \mathcal{E}^+$ is isomorphic to the trivial bundle over X;*
   - *the following diagram commutes*

$$\begin{tikzcd} E^- \arrow[r, "\Pi"] \arrow[d, "\pi^-"'] & E^+ \arrow[d, "\pi^+"] \\ X \arrow[r, "\theta"] & X \end{tikzcd}$$

   - *for every $k \in X$, the map $\Pi_k : \mathcal{E}_k^- \to \mathcal{E}_{\theta(k)}^+$ is anti-unitary and satisfies $\Pi_k^* \circ \Pi_k = \pm 1_{\mathcal{E}_k^-}$.*
3. *Chiral symmetric bundle. A pair of $\mathbb{K}$-vector bundles $\mathcal{E}^- = (E^-, \pi^-, X)$ and $\mathcal{E}^+ = (E^+, \pi^+, X)$ over X are chiral symmetric if there exists a map $\Sigma : \mathcal{E}^- \to \mathcal{E}^+$ such that:*
   - *the direct sum $\mathcal{E}^- \oplus \mathcal{E}^+$ is isomorphic to the trivial bundle over X;*
   - *the following diagram commutes*

$$\begin{tikzcd} E^- \arrow[r, "\Sigma"] \arrow[d, "\pi^-"'] & E^+ \arrow[d, "\pi^+"] \\ X \arrow[r, "1_X"] & X \end{tikzcd}$$

   - *for every $k \in X$, the map $\Sigma_k : \mathcal{E}_k^- \to \mathcal{E}_k^+$ is unitary and satisfies $\Sigma_k^* \circ \Sigma_k = 1_{\mathcal{E}_k^-}$.*

These symmetric bundles serve as the foundation for the classification of topological insulators under the so-called "ten-fold way" whose name takes inspiration both from the "eight-fold way" and the the "three-fold way". The first one is an organizational scheme for a class of subatomic particles known as baryons and mesons that led to the development of the quark model [45, 46]. The classification scheme is based on the internal symmetry group SU(3) of the Standard Model (the QCD sector) and specifically on the roots system of its Lie algebra. It organizes particles into multiplets according to their isospin[6] and strangeness quantum numbers, reflecting the underlying flavor symmetries of the strong interaction. In its more general declination, the classification scheme is based on the flavor symmetry SU(6) and its Lie algebra root system which encodes all the quark possibilities: up, down, strange, bottom, charm and beauty. This framework successfully predicted the existence of previously unobserved particles, such as the Ω^- baryon. The second one is a classification scheme for the gaussian random matrix models which have a lot of application in Physics [47, 48]. The classification identifies three models: Gaussian Unitary Ensamble (GUE), Gaussian Orthogonal Ensamble (GOE) and Gaussian Symplectic Ensamble (GSE).
In contrast, the "ten-fold way" is a modern classification of condensed matter quantum systems, especially topological insulators and superconductors, based on discrete and translational symmetries in quantum mechanics. Both frameworks provide powerful organizational principles, but they differ in scope, symmetry structure, and physical context.

Each type of symmetric Bloch bundle induces specific K-theoretic structures, leading to a twisted equivariant K-theory. Therefore, let us briefly discuss what "twisted" and "equivariant" means in the case of K-theory [49, 50, 51, 52].
On the one hand, twisted K-theory is a generalization of K-theory, where we consider a twist, i.e. an additional piece of data that modifies the definition of $\mathbb{K}$-vector bundles and therefore changes the resulting K-theory groups. A prototypical example of twist is given by a class $\tau \in H^3(X;\mathbb{Z})$. This class represents an obstruction to the existence of globally defined ordinary $\mathbb{K}$-vector bundles. More precisely we can consider projective bundles, i.e. bundles that are locally biholomorphic to a projective space $\mathbb{P}^n\mathbb{C}$ instead of $\mathbb{C}^n$ and the twist τ represents the obstruction to lifting a projective bundle to a complex vector bundle. In practical terms, we are doing K-theory not of vector bundles but of modules over more complicated "twisted" algebras determined by the class τ. For example, in physical application beyond Condensed Matter Physics, such as in String Theory, the twist $\tau \in H^3(X;\mathbb{Z})$ corresponds to the field strength of the so-called B-field (or Kalb-Ramond field), that is a closed 3-form appearing in the Lagrangian formulation of the string. As a simple example where K-theory can be enhanced to its twisted version is the case of $X = S^3$, the 3-sphere. Then $H^3(S^3;\mathbb{Z}) \simeq \mathbb{Z}$, so every integer defines a possible twist.
On the other hand, equivariant cohomology is a tool that allows us to study the topology of a space X together with the action of a group G on it. It generalizes ordinary cohomology by incorporating the symmetry of the space given by the group

[6] In the simplest case in which only the quark up, down and strange are considered the isospin is given by the Gell Mann-Nishijima law: $q = I_z + Y/2$ where I_z is the z-component projection of the isospin and Y is the hypercharge given by the sum of barionic and strange flavor numbers.

action. Indeed, while ordinary cohomology $H^\bullet(X)$ captures topological features it ignores any symmetry encoded by the group action. Equivariant cohomology $H^\bullet_{\mathrm{G}}(X)$ is designed to reflect both the topology of X and the way G acts on it. The most common way to define equivariant cohomology is through the so-called Borel construction: we start with the contractible space $E\mathrm{G}$ of Definition E.0.3 on which G acts freely and then we define the

$$X_{\mathrm{G}} := (E\mathrm{G} \times X)/\mathrm{G},$$

where G acts diagonally on $E\mathrm{G} \times X$; the equivariant cohomology of X is then defined as the ordinary cohomology of X_{G}

$$H^\bullet_{\mathrm{G}}(X) := H^\bullet(X_{\mathrm{G}}).$$

As we saw around Definition E.0.3, the space $E\mathrm{G}$ always exists for nice enough groups such as Lie groups.

The classification of topological insulators consists in the tabulation of the reduced K^0-groups of symmetric Bloch bundles over spheres where the relevant involution on S^d is the one induced by the involution $k \mapsto -k$ on $\mathbb{R}^d$ under the one-point compactification. By Bott Theorem 3.1.2, in the complex symmetry classes, those lacking any symmetry or possessing only a chiral symmetry, a 2-fold periodicity in the spatial dimension emerges. In contrast, the real symmetry classes, which involve anti-unitary symmetries such as time reversal or particle hole conjugation, exhibit an 8-fold periodicity in the dimension. The following table reports the classification.

Table 3.3. Periodic table of topological insulators [53]. The rows label symmetry classes following the Altland-Zirnbauer classification, the columns T, C, S show the presence or not of the respective discrete symmetry showing the value of its square (+1 if it is $+1_{\mathcal{H}}$, -1 if it is $-1_{\mathcal{H}}$) while the last eight columns label the spatial dimensions. The entries show the reduced K^0-groups of appropriate symmetric vector bundles over spheres in each symmetry class and dimension.

Symmetry Class	T	C	S	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

In the next Chapter, our attention will move on to systems that have all three discrete symmetries. However, we are interested in a classification in the sense of isomorphism of Bloch bundles rather than a K-theoretical classification. The reasons are the following:

- we are interested in investigating low dimensional cases which are cases where explicit calculations are feasible and so the homotopy classification is an achievable goal;
- the K-theoretical approach is more powerful but also coarser; it essentially takes into account invariants that are actually related to the dimension under consideration while it is possible that there exist topological invariants related to lower dimensional parts of the system under consideration, the so-called weak invariants. The object useful to describe weak invariants of 1D subsystems is the Berry phase, hence we are interested in studying this invariant in 1D systems that exhibit all three discrete symmetries we have discussed.

Chapter 4

Berry phase as topological invariants for BDI, DIII, CII, CI chains

The study of topological insulators has emerged as a central theme at the intersection of Condensed Matter Physics and modern mathematics, due both to their potential for novel technological applications and to the rich mathematical framework underlying their classification. Physically, a topological insulator is a quantum system described by a self-adjoint Hamiltonian operator H acting on a Hilbert space $\mathcal{H}$, such that its energy spectrum is gapped. This gap allows for a well-defined separation between the occupied and unoccupied energy bands. As we saw, when the system enjoys translational invariance, its analysis can be recast in momentum space via the Zak-Bloch-Floquet transform. In this setting, the Hilbert space becomes fibered over the Brillouin torus. The occupied states define a subbundle whose topological features classify the phase of the material. Characteristic classes such as the Chern classes and elements of K-theory, serve as invariants for these vector bundles and encode the topological information of the phase. A cornerstone of the theory is the bulk-boundary correspondence, which posits that the non-trivial topology of the bulk Hamiltonian implies the existence of robust, gapless boundary modes when the system is spatially truncated. This point has multiple technological applications. The edge states are protected by the topological nature of the bulk and remain stable under local perturbations that do not close the spectral gap or break the relevant symmetries. The interplay between geometry, topology, and quantum physics in this field has led to new developments in mathematical physics, including the use of tools from non-commutative geometry, index theory and equivariant K-theory; moreover, the efforts to understand the mathematics underlying topological insulators make these materials more controllable and more applicable to technological applications.

The main goal of this Chapter, based on original results that extend those of [54], is to investigate how much the Berry phase can tell on the topology classification of 1D topological insulators that shows all the three discrete symmetries we discussed: time reversal, charge conjugation and chiral symmetry. These symmetry classes, referring to Table 3.3, are: BDI, DIII, CII, CI.

4.1 Set up and preliminaries

We are interested in 1D systems but we will discuss everything in arbitrary dimension d until otherwise specified. We require the systems to enjoy translational invariance under a lattice Γ of the full $(\mathbb{R}^d, +)$ group. Therefore we assume a self-adjoint Hamiltonian such that $[U_\lambda, H] = 0$ for all $\lambda \in \Gamma$. Moreover, we assume that the spectrum of H has a spectral gap around zero, hence there exists $g > 0$ such that $(-g, g) \cap \sigma(H) = \emptyset$. We are describing a topological insulator according to Definition 3.3.1. The Hamiltonian of the system can be decomposed, according to the general theory discussed in Subsections 2.2.1 and 2.2.2, in fibered Hamiltonians H_k. On top on that, we require these systems enjoy also time reversal, particle hole and chiral symmetry, see Paragraph 3.2.

4.1.1 The action of the symmetry operators

Since we have all the discrete symmetries of Section 3.2 we have, due to the commutativity of different symmetry operators, seven possibilities of symmetry operators that can act on vectors of the Hilbert space of the system: T, C, S, TC, TS, CS, TCS. In a rigorous way we can show the following general result.

Theorem 4.1.1 (Group generated by discrete symmetry operators T, C, S)**.** *Let T, C, S be operators on a Hilbert space $\mathcal{H}$ such that:*

1. *T, C are anti-unitary and S is unitary;*
2. *each is involutive up to a sign, i.e., $T^2, C^2, S^2 \in \{\pm 1_{\mathcal{H}}\}$;*
3. *T, C, S commute pairwise.*

Then the group $(\mathrm{G}, \circ)$ generated by T, C, S, modulo the subgroup of global sign factors $\{\pm 1_{\mathcal{H}}\}$, is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. That is,

$$\mathrm{G}/\{\pm 1_{\mathcal{H}}\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Proof. Let $(\mathrm{G}, \circ) \subset (\mathrm{Aut}(\mathcal{H}), \circ)$ be the group generated by T, C, S with composition of operators as group multiplication (we will omit the symbol as done until now). Since T, C, S are involutive up to sign, we have

$$T^2 = \pm 1_{\mathcal{H}}, \quad C^2 = \pm 1_{\mathcal{H}}, \quad S^2 = \pm 1_{\mathcal{H}},$$

so each generator satisfies $X^4 = 1$. Moreover, since the operators commute,

$$TC = CT, \quad TS = ST, \quad CS = SC;$$

the group $(\mathrm{G}, \circ)$ is abelian. Now consider the quotient

$$\widetilde{\mathrm{G}} := \mathrm{G}/\{\pm 1_{\mathcal{H}}\}.$$

In $\widetilde{\mathrm{G}}$, two operators are identified if they differ only by a global sign.
We construct the elements of $\widetilde{\mathrm{G}}$ explicitly. Each element of G can be written as a product of T, C, S, namely:

$$1_{\mathcal{H}}, \ T, \ C, \ S, \ TC, \ TS, \ CS, \ TCS.$$

Since T, C, S are involutive up to sign and commute pairwise, these 8 elements are distinct in $\widetilde{\mathrm{G}}$ and represent all cosets modulo sign. To show the isomorphism we define a map:

$$\Phi : \widetilde{\mathrm{G}} \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad [T^a C^b S^c] \mapsto (a, b, c)$$

for $a, b, c \in \{0, 1\}$.

This map is a group isomorphism since:

1. it is well-defined as each coset has a unique representative of the form $T^a C^b S^c$;
2. it is surjective since every binary triple has a preimage;
3. it is injective: in fact, different equivalence classes have different binary triples;
4. it is a homomorphism, since multiplication in $\widetilde{\mathrm{G}}$ corresponds to addition modulo 2 in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Therefore:

$$\widetilde{\mathrm{G}} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

□

The above result can be further generalized to cases in which some or all of the discrete symmetry operators anti-commute. In these cases we can show that $\tilde{\mathrm{G}}$ is no more abelian and turns out to be a subgroup of the multiplicative group of the Clifford algebra. The interesting point of the above result is that under the hypothesis of commutativity the resulting algebraic structure is the one of an abelian finite group.

Let us now study the action of the discrete symmetry operators on the fibered Hamiltonians regardless of the commutativity properties of the operators. Let $v \in \mathbb{C}^N$ be an eigenvector of the fiber Hamiltonian H_k with eigenvalue E, that is,

$$H_k(v) = Ev. \tag{4.1}$$

We examine the effect of various symmetry operators on v, and compute the resulting eigenvalue equations using the results of Proposition 3.3.1

Time reversal symmetry T

We compute

$$H_{-k}(\mathfrak{T}(v)) = \mathfrak{T}(H_k(v)) = \mathfrak{T}(Ev) = E\mathfrak{T}(v). \tag{4.2}$$

Thus, $\mathfrak{T}(v)$ is an eigenvector of H_{-k} with eigenvalue E.

Particle hole symmetry C

We compute

$$H_{-k}(\mathfrak{C}(v)) = -\mathfrak{C}(H_k(v)) = -\mathfrak{C}(Ev) = -E\mathfrak{C}(v). \tag{4.3}$$

Thus, $\mathfrak{C}(v)$ is an eigenvector of H_{-k} with eigenvalue $-E$.

Chiral symmetry S

We compute

$$H_k(\mathfrak{S}(v)) = -\mathfrak{S}(H_k(v)) = -\mathfrak{S}(Ev) = -E\mathfrak{S}(v). \tag{4.4}$$

Thus, $\mathfrak{S}(v)$ is an eigenvector of H_k with eigenvalue $-E$.

Combined symmetry TC

We compute

$$H_k(\mathfrak{T}(\mathfrak{C}(v))) = -\mathfrak{T}(\mathfrak{C}(H_k(v))) = -\mathfrak{T}(\mathfrak{C}(Ev)) = -E\mathfrak{T}(\mathfrak{C}(v)). \tag{4.5}$$

Thus, $\mathfrak{T}(\mathfrak{C}(v))$ is an eigenvector of H_k with eigenvalue $-E$.

Combined symmetry TS

We compute

$$H_{-k}\mathfrak{T}(\mathfrak{S}(v))) = -\mathfrak{T}(\mathfrak{S}(H_k(v))) = -\mathfrak{T}(\mathfrak{S}(Ev)) = -E\mathfrak{T}(\mathfrak{S}(v)). \tag{4.6}$$

Thus, $\mathfrak{T}(\mathfrak{S}(v))$ is an eigenvector of H_{-k} with eigenvalue $-E$.

Combined symmetry CS

We compute

$$H_{-k}(\mathfrak{C}(\mathfrak{S}(v))) = \mathfrak{C}(\mathfrak{S}(H_k(v))) = \mathfrak{C}(\mathfrak{S}(Ev)) = E\mathfrak{C}(\mathfrak{S}(v)). \tag{4.7}$$

Thus, $\mathfrak{C}(\mathfrak{S}(v))$ is an eigenvector of H_{-k} with eigenvalue E.

Combined symmetry TCS

We compute

$$H_k(\mathfrak{T}(\mathfrak{C}(\mathfrak{S}(v)))) = \mathfrak{T}(\mathfrak{C}(\mathfrak{S}(H_k(v)))) = \mathfrak{T}(\mathfrak{C}(\mathfrak{S}(Ev))) = E\mathfrak{T}(\mathfrak{C}(\mathfrak{S}(v))). \tag{4.8}$$

Thus, $\mathfrak{T}(\mathfrak{C}(\mathfrak{S}(v)))$ is an eigenvector of H_k with eigenvalue E.

The following table summarize all the results.

Table 4.1. Summary of the symmetry operators $\mathfrak{T}$, $\mathfrak{C}$, $\mathfrak{S}$ and their possible combinations. We report their actions on quasi-momentum k and on the fiber Hamiltonian H_k. The last column shows the resulting spectral symmetry.

Operator	**Action on k**	**Action on H_k**	**Spectral Symmetry**
$\mathfrak{T}$	$k \mapsto -k$	$H_{-k} = \mathfrak{T} H_k \mathfrak{T}^{-1}$	$E \leftrightarrow E$
$\mathfrak{C}$	$k \mapsto -k$	$H_{-k} = -\mathfrak{C} H_k \mathfrak{C}^{-1}$	$E \leftrightarrow -E$
$\mathfrak{S}$	$k \mapsto k$	$H_k = -\mathfrak{S} H_k \mathfrak{S}^{-1}$	$E \leftrightarrow -E$
$\mathfrak{TC}$	$k \mapsto k$	$H_k = -(\mathfrak{TC}) H_k (\mathfrak{TC})^{-1}$	$E \leftrightarrow -E$
$\mathfrak{TS}$	$k \mapsto -k$	$H_{-k} = -(\mathfrak{TS}) H_k (\mathfrak{TS})^{-1}$	$E \leftrightarrow -E$
$\mathfrak{CS}$	$k \mapsto -k$	$H_{-k} = (\mathfrak{CS}) H_k (\mathfrak{CS})^{-1}$	$E \leftrightarrow E$
$\mathfrak{TCS}$	$k \mapsto k$	$H_k = (\mathfrak{TCS}) H_k (\mathfrak{TCS})^{-1}$	$E \leftrightarrow E$

The above computations show that in systems BDI, DIII, CII and CI there are always symmetries with a non-trivial action on the spectrum: particle hole symmetry, chiral symmetry and their combinations with time reversal symmetry map E to $-E$. According to the theory of decomposable operators outlined in Subsections 2.2.1 and 2.2.2, the spectrum of the original Hamiltonian H is given by the union of the spectra of the fiber Hamiltonians H_k with $k \in \mathbb{T}^d$. Therefore, $\sigma(H)$ must be symmetric with respect to $E = 0$ since if $\lambda \in \sigma(H)$ then $-\lambda \in \sigma(H)$ as well. In Quantum Mechanics, and more generally in Physics, it is customary to impose a stability condition on the energy operator, namely that H is bounded from below (see Definition 1.2.4). Under such a condition, the presence of spectral symmetries forces the Hamiltonian to be bounded, making the use of continuum Schrödinger-type operators on a space of square integrable functions not possible since these kind of operators are not bounded due to the kinetic term. In view of the spectral symmetry and the boundedness condition discussed above, we shall restrict our attention to tight-binding models. Consequently, we assume that the configuration space of the quantum system is modeled on a dD discrete lattice $\mathbb{Z}^d$. To account for additional internal degrees of freedom, such as spin, orbital states, or sublattice structure, we introduce N internal components per lattice site. The one-particle Hilbert space is thus taken to be

$$\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N, \tag{4.9}$$

where $\ell^2(\mathbb{Z}^d)$ encodes the spatial structure of the lattice and $\mathbb{C}^N$ captures the internal degrees of freedom at each site. In these notations, the discrete symmetry operators acts as the identity on $\ell^2(\mathbb{Z}^d)$ and we call T, C, S the operators acting non-trivially on the $\mathbb{C}^N$-leg of the tensor product Hilbert space $\mathcal{H}$.
We denote by $U_\lambda : \mathcal{H} \to \mathcal{H}$, with $\lambda \in \mathbb{Z}^d$, the unitary translation operators defined by

$$(U_\lambda(\psi))_n := \psi_{n+\lambda}, \quad \text{for } \psi = (\psi_n) \in \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N. \tag{4.10}$$

It is a standard result that a bounded, self-adjoint, translation-invariant operator H on $\mathcal{H}$ must act via convolution in the lattice variable. That is, for any $\psi = (\psi_n) \in \mathcal{H}$, the action of H can be written as

$$(H(\psi))_n = A_0\psi_n + \sum_{j\in\mathbb{N}^d\setminus\{0\}} \left(A_j\psi_{n+j} + A_j^*\psi_{n-j}\right), \tag{4.11}$$

for an appropriate family of matrices $A_j \in \mathrm{M}_N(\mathbb{C})$, indexed by $j \in \mathbb{N}^d$, with $A_0 = A_0^*$. Alternatively, if we denote by $\{e_n\}_{n\in\mathbb{Z}^d}$ the canonical basis of $\ell^2(\mathbb{Z}^d)$ and let $v \in \mathbb{C}^N$, then the action of H on elementary tensors can be expressed as

$$H(e_n \otimes v) = e_n \otimes A_0 v + \sum_{j\in\mathbb{N}^d\setminus\{0\}} \left(e_{n+j} \otimes A_j v + e_{n-j} \otimes A_j^* v\right), \tag{4.12}$$

and extended to all of $\mathcal{H}$ by linearity and continuity. Physically, each matrix A_j can be interpreted as a matrix-valued hopping amplitude for a particle to move by a vector displacement j on the lattice.
Boundedness of H imposes decay conditions on the operator norms $\|A_j\|$, typically requiring them to vanish at infinity. To simplify the setting, we shall restrict ourselves to the case of finite-range operators, where hopping occurs only within a bounded neighborhood. This is physically motivated since in real systems such as crystalline solids, electrons move on an atomic lattice and the probability of hopping from one site to another decays rapidly with distance due to the spatial localization of LCAO (Linear Combination of Atomic Orbitals) and the screening of interactions [19, 55, 56, 57]. Indeed, the hopping matrices A_j typically decay as

$$\|A_j\| \sim \mathcal{O}(e^{-\alpha|j|_{\ell^1}}), \tag{4.13}$$

with $\alpha > 0$, so contributions from distant hopping decay exponentially and become negligible for practical purposes. Moreover, the rapid decay of long-range interactions is also due to the screening of Coulomb interactions by the surrounding electronic cloud. In a many electron system, the effective interaction between charges is given by a screened Coulomb potential, i.e. a Yukawa-type potential,

$$V(r) \propto \frac{e^{-\kappa r}}{r}, \tag{4.14}$$

where κ^{-1} is the screening length, determined by the material's electronic properties; κ can be viewed as an effective mass parameter for the electric field in the medium. From a modeling perspective, this justifies including only nearest or next-to-nearest neighbors. Moreover, this approximation preserves the local symmetries of the lattice and significantly simplifies the mathematical analysis, without affecting the system's topological or phase properties. Finally, in the thermodynamic and low energy limits, long range hopping contributions are negligible for many physically relevant observables [58, 59]. Therefore, we assume that there exists $R \in \mathbb{N}$ such that

$$A_j = 0 \qquad \forall\, j \in \mathbb{N}^d \setminus \{0\} \mid |j|_{\ell^1} > R. \tag{4.15}$$

4.1.2 Explicit writing of the fiber Hamiltonians and spectral eigenprojectors

Before moving on we need the following result on L^2-spaces; the full proof in the general case can be found in [60].

Theorem 4.1.2 (Tensor product isomorphism for L^2-spaces)**.** *Let (X, μ) and (Y, ν) be σ-finite measure spaces. Then there exists a natural unitary isomorphism of Hilbert spaces*

$$L^2(X \times Y, \mu \times \nu) \simeq L^2(X, \mu) \otimes L^2(Y, \nu),$$

where $\otimes$ denotes the completed Hilbert space tensor product. In particular, if (S^1, μ) is the unit circle with normalized Lebesgue measure, then for every $d \in \mathbb{N}$:

$$L^2\left((S^1)^{\times d}\right) \simeq \bigotimes_{j=1}^{d} L^2(S^1),$$

with orthonormal basis given by $\frac{e^{in\cdot\theta}}{(\sqrt{2\pi})^d}$ for $n \in \mathbb{Z}^d$ and $\theta \in (S^1)^{\times d}$.

Proof. Let $\mathcal{H}_1 = L^2(X, \mu)$ and $\mathcal{H}_2 = L^2(Y, \nu)$. We define the algebraic tensor product $\mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2$ as the set of finite linear combinations of elements $f \otimes g$, with $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$. We introduce an inner product on simple tensors by defining

$$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle := \langle f_1, f_2 \rangle_{\mathcal{H}_1} \langle g_1, g_2 \rangle_{\mathcal{H}_2},$$

and we extend it sesquilinearly to $\mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2$. The completion with respect to the norm induced by this inner product is the Hilbert space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Now, we define a linear map $\Phi : \mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2 \to L^2(X \times Y, \mu \times \nu)$ by

$$\Phi\left(\sum_{j=1}^{n} f_j \otimes g_j\right)(x, y) := \sum_{j=1}^{n} f_j(x) g_j(y).$$

We first note that the map is well-defined. Indeed, by Fubini/Tonelli's theorem, each term $f_j(x)g_j(y)$ is square integrable, so Φ maps into $L^2(X \times Y)$. Moreover, this is an isometry. The norm of the image is

$$\left\| \Phi\left(\sum_{j=1}^{n} f_j \otimes g_j\right) \right\|_{L^2(X \times Y)}^2 = \int_{X \times Y} \left| \sum_{j=1}^{n} f_j(x) g_j(y) \right|^2 d\mu(x) d\nu(y),$$

and expanding the square modulus we have

$$\left| \sum_{j=1}^{n} f_j(x) g_j(y) \right|^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} f_j(x) \overline{f_k(x)} g_j(y) \overline{g_k(y)};$$

therefore, using Fubini/Tonelli's theorem,

$$\begin{aligned} \left\| \Phi\left(\sum_{j=1}^{n} f_j \otimes g_j\right) \right\|^2 &= \int_{X \times Y} \sum_{j,k=1}^{n} f_j(x) \overline{f_k(x)} g_j(y) \overline{g_k(y)} d\mu(x) d\nu(y) = \\ &= \sum_{j,k=1}^{n} \left(\int_X f_j(x) \overline{f_k(x)} d\mu(x) \right) \left(\int_Y g_j(y) \overline{g_k(y)} d\nu(y) \right) = \\ &= \sum_{j,k=1}^{n} \langle f_j, f_k \rangle_{\mathcal{H}_1} \langle g_j, g_k \rangle_{\mathcal{H}_2} = \left\| \sum_{j=1}^{n} f_j \otimes g_j \right\|_{\mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2}^2 . \end{aligned}$$

Hence, the map Φ is an isometry. At this point, since the image of Φ contains all finite sums of separable functions $f(x)g(y)$, which are dense in $L^2(X \times Y)$ by standard results in L^2 theory, Φ extends to a unitary isomorphism between $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $L^2(X \times Y)$.
In particular, let $L^2(S^1)$ denote the square integrable functions on the unit circle with normalized Lebesgue measure; then $L^2(S^1)$ has orthonormal basis $\left\{\frac{e^{in\theta}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$.
The tensor product $\bigotimes_{j=1}^d L_j^2(S^1)$ is spanned by the functions of the form

$$\frac{e^{i(n_1\theta_1)}}{\sqrt{2\pi}} \otimes \frac{e^{i(n_2\theta_2)}}{\sqrt{2\pi}} \otimes \cdots \otimes \frac{e^{i(n_d\theta_d)}}{\sqrt{2\pi}},$$

for $(n_1, \ldots, n_d) \in \mathbb{Z}^d$. The map Φ maps finite combination of the above functions in finite combination of function of the form

$$\frac{e^{in\cdot\theta}}{(\sqrt{2\pi})^d} := \frac{e^{i(n_1\theta_1+\cdots+n_d\theta_d)}}{(\sqrt{2\pi})^d}, \qquad n := (n_1, \ldots, n_d) \in \mathbb{Z}^d, \theta := (\theta_1, \ldots, \theta_d) \in (S^1)^{\times d},$$

that forms an orthonormal basis for $L^2((S^1)^{\times d})$, providing the desired isomorphism $L^2((S^1)^{\times d}) \simeq \bigotimes_{j=1}^d L_j^2(S^1)$. □

The transform that decomposes the Hamiltonian in fiber operators, which is the Bloch-Floquet transform (essentially a vector valued Fourier transform) is defined explicitly by

$$\mathcal{F}_d : \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \to L^2((S^1)^{\times d}; \mathbb{C}^N), \quad \mathcal{F}_d(e_n \otimes v) := \frac{e^{in\cdot k}}{(\sqrt{2\pi})^d} v, \qquad n \in \mathbb{Z}^d, v \in \mathbb{C}^N. \tag{4.16}$$

and this is a bijective linear isometry because it sends an orthonormal basis to another orthonormal basis. We can now state the following result.

Proposition 4.1.1 (Fiber Hamiltonians). *The operator $\mathcal{F}_d H \mathcal{F}_d^{-1}$ is a fibered operator whose fiber Hamiltonians are matrices $H_k \in \mathrm{M}_N(\mathbb{C})$ given by*

$$H_k = A_0 + \sum_{j\in J} \left(e^{-ij\cdot k} A_j + e^{ij\cdot k} A_j^*\right), \qquad k \in \mathbb{T}^d,$$

where $J := \{j \in \mathbb{N}^d \setminus \{0\} \mid |j|_{\ell^1} \leq R\}$.

Proof. Using definition (4.16), condition (4.15) and the action of the Hamiltonian (4.12) we have

$$\mathcal{F}_d H \mathcal{F}_d^{-1} \left(\frac{e^{in\cdot k}}{(\sqrt{2\pi})^d} \otimes v\right) = \mathcal{F}_d \left[e_n \otimes A_0 v + \sum_{j\in J} \left(e_{n-j} \otimes A_j v + e_{n+j} \otimes A_j^* v\right)\right] =$$

$$= \frac{e^{in\cdot k}}{(\sqrt{2\pi})^d} \otimes A_0 v + \sum_{j\in J} \left[\frac{e^{i(n-j)\cdot k}}{(\sqrt{2\pi})^d} \otimes A_j v + \frac{e^{i(n+j)\cdot k}}{(\sqrt{2\pi})^d} \otimes A_j^* v\right] =$$

$$= \left[A_0 + \sum_{j=1}^{R} \left(e^{-ij\cdot k} A_j + e^{ij\cdot k} A_j^*\right)\right] \left(\frac{e^{in\cdot k}}{(\sqrt{2\pi})^d} \otimes v\right),$$

where we used that $L^2((S^1)^{\times d};\mathbb{C}^N) \simeq L^2((S^1)^{\times d}) \otimes \mathbb{C}^N$. Since the functions

$$\left\{ \frac{e^{in\cdot k}}{(\sqrt{2\pi})^d} \right\}_{n\in\mathbb{Z}^d}$$

form an form an orthonormal basis of $L^2((S^1)^{\times d})$, the conclusion follows by linearity. □

We observe that the quasi-momentum $k = 0$ is fixed under the involution $\Phi : k \mapsto -k$. Therefore, by the spectral theorem and the considerations above, $\mathbb{C}^N$ admits an orthonormal basis $v_1, \ldots, v_N$ consisting of eigenvectors of the self-adjoint matrix H_0 assuming a spectral gap around $\mu = 0$. Among these, the first m vectors $v_1, \ldots, v_m$ correspond to negative eigenvalues, while the remaining $v_{m+1}, \ldots, v_N$ correspond to positive eigenvalues. The presence of at least one symmetry operator with spectral symmetry $E \leftrightarrow -E$, imposes the existence of a bijection between the eigenspaces of H_0 associated with negative and positive eigenvalues. As a consequence, it must be that $m = N - m$, and hence the fiber dimension N must be even, $N = 2m$.

By the general theory and the explicit representation of Proposition 4.1.1, H_k depends analytically on $k \in \mathbb{T}^d$ and it is 2π-periodic in each component of k. Thanks to the spectral gap hypothesis, the negative and positive eigenspaces of H_k depend analytically on $k \in \mathbb{T}^d$ and inherit the 2π-periodicity in each component of k. To define the spectal eigenprojectors we employ Riesz formula (2.31). Let $r > 0$ be such that $-2r$ is a lower bound for the spectrum of H, and hence uniformly for the spectra of H_k with $k \in \mathbb{T}^d$. We have

$$P_k^{(-)} := \frac{i}{2\pi} \int_{\partial B_r(-r)} (H_k - z1_{\mathbb{C}^N})^{-1} dz, \qquad k \in \mathbb{T}^d, \tag{4.17}$$

where $\partial B_r(-r)$ denotes the positively oriented circle in the complex plane centered at $-r$ with radius r. This contour encloses only the negative part of the spectrum of H_k, where the resolvent $(H_k - z1_{\mathbb{C}^N})^{-1}$ has its poles. We note that the integration in (4.17) is performed in the complex energy plane, not in momentum space; moreover the matrix-valued integral is understood entry-wise. The operator $P_k^{(-)}$ defined by Riesz formula (4.17) is an orthogonal eigenprojection that projects onto the direct sum of eigenspaces associated with the negative eigenvalues of H_k. By functional calculus, it follows that $P_k^{(-)}$ inherits the analytic and periodic properties of H_k with respect to $k \in \mathbb{T}^d$. The complementary spectral eigenprojection onto the positive part of the spectrum is given by

$$P_k^{(+)} := 1_{\mathbb{C}^N} - P_k^{(-)}. \tag{4.18}$$

4.2 Symmetric Bloch bases

We are interested in quantifying the topological content of the spectral eigenprojections of the fiber operators H_k into a numerical invariant. To do so we start by identifying bases with particular properties under the discrete symmetries of the system.

Definition 4.2.1 (Bloch basis). *Given a family of rank-m projectors* $\{P_k\} \in \mathrm{M}_N(\mathbb{C})$, *which is* 2π*-periodic in each component of* k *and at least* C^1 *in* k, *we define* $\{v_i(k)\}_{i=1}^N \in \mathbb{C}^N$ *to be a Bloch basis associated to* $\{P_k\}$ *if it satisfies the following properties for every* $k \in \mathbb{R}^d$*:*

1. *the collection* $\{v_i(k)\}_{i=1}^N$ *forms an orthonormal basis of* $\mathbb{C}^N$*;*
2. *the vectors* $\{v_i(k)\}_{i=1}^m$ *span* $\mathrm{Range}(P_k)$ *and therefore the vectors* $\{v_i(k)\}_{i=m+1}^N$ *span* $\mathrm{Ker}(P_k) = \mathrm{Range}(1_{\mathbb{C}^N} - P_k)$*;*
3. *each vector* $v_i(k)$ *is* 2π*-periodic and as regular as* P_k *as a function of* $k \in \mathbb{R}^d$.

Since the systems under considerations enjoy some discrete symmetries, we want a Bloch basis to have well-defined transformation properties under the discrete symmetry operators of the system. The transformation properties are inferred from Table 4.1. Therefore we give the following Definition.

Definition 4.2.2 (Symmetric Bloch basis). *Let* $P_k^{(-)}$ *and* $P_k^{(+)}$ *be as in equations* (4.17) *and* (4.18). *Let* $N = 2m$, *hence both* $P^-(k)$ *and* $P^+(k)$ *have rank* m. *Then, a Bloch basis* $\{v_i(k)\}_{i=1}^N$ *associated to* $\left\{P_k^{(-)}\right\}$ *is said to be a symmetric Bloch basis if the following are satisfied:*

1. *let* $\mathfrak{O}^{(++)} \in \{\mathfrak{T}\mathfrak{C}\mathfrak{S}\}$*; then*

$$\mathfrak{O}^{(++)}\, v_i(k) = v_i(k), \qquad \forall\ i \in \{1, ..., 2m\} \quad \text{if} \quad \left(\mathfrak{O}^{(++)}\right)^2 = +1_{\mathbb{C}^{2m}};$$
$$\mathfrak{O}^{(++)}\, v_i(k) = -\sum_{j=1}^{m} v_j(k)\varepsilon_{ij}, \qquad \forall\ i \in \{1, ..., m\} \quad \text{if} \quad \left(\mathfrak{O}^{(++)}\right)^2 = -1_{\mathbb{C}^{2m}};$$

2. *let* $\mathfrak{O}^{(+-)} \in \{\mathfrak{S}, \mathfrak{T}\mathfrak{C}\}$*; then*

$$\mathfrak{O}^{(+-)}\, v_i(k) = v_{N-i+1}(k), \qquad \forall\ i \in \{1, ..., m\};$$

3. *let* $\mathfrak{O}^{(-+)} \in \{\mathfrak{T}, \mathfrak{C}\mathfrak{S}\}$*; then*

$$\mathfrak{O}^{(-+)}\, v_i(k) = v_i(-k), \qquad \forall\ i \in \{1, ..., 2m\} \quad \text{if} \quad \left(\mathfrak{O}^{(-+)}\right)^2 = +1_{\mathbb{C}^{2m}};$$
$$\mathfrak{O}^{(-+)}\, v_i(k) = -\sum_{j=1}^{m} v_j(-k)\varepsilon_{ij}, \qquad \forall\ i \in \{1, ..., m\} \quad \text{if} \quad \left(\mathfrak{O}^{(-+)}\right)^2 = -1_{\mathbb{C}^{2m}};$$

4. *let* $\mathfrak{O}^{(--)} \in \{\mathfrak{C}, \mathfrak{T}\mathfrak{S}\}$*; then*

$$\mathfrak{O}^{(--)}\, v_i(k) = v_{N-i+1}(-k), \qquad \forall\ i \in \{1, ..., m\};$$

where ε_{ij} *are reshuffling matrices which are unitary and anti-symmetric.*

The necessity of reshuffling matrices is evident if one looks at the invariant subspace $k = 0$. In fact, it is possible to build a symplectic form such that the vectors $\mathfrak{O}^{(++)}v(0)$ and/or $\mathfrak{O}^{(-+)}v(0)$ are orthogonal to $v(0)$ if the operators square

to $-1_{\mathbb{C}^{2m}}$. In particular this would imply the vanishing of the vectors and prevents from setting the definition used for operators that square to $+1_{\mathbb{C}^{2m}}$ as showed in [61, 62].
Again from Table 4.1, we can deduce the action of the discrete operators on the spectral eigenprojectors. These are summarized by the following relations.

Proposition 4.2.1 (Discrete symmetries and spectral eigenprojectors)**.** *Let* $P_k^{(-)}$ *and* $P_k^{(+)}$ *be as in equations* (4.17) *and* (4.18)*, then*

1. *let* $\mathfrak{O}^{(++)} \in \{\mathfrak{TCS}\}$ *then*
$$\mathfrak{O}^{(++)} P_k^{(+/-)} = P_k^{(+/-)} \mathfrak{O}^{(++)};$$
2. *let* $\mathfrak{O}^{(+-)} \in \{\mathfrak{S}, \mathfrak{TC}\}$ *then*
$$\mathfrak{O}^{(+-)} P_k^{(+)} = P_k^{(-)} \mathfrak{O}^{(+-)};$$
3. *let* $\mathfrak{O}^{(-+)} \in \{\mathfrak{T}, \mathfrak{CS}\}$ *then*
$$\mathfrak{O}^{(-+)} P_k^{(+/-)} = P_{-k}^{(+/-)} \mathfrak{O}^{(-+)};$$
4. *let* $\mathfrak{O}^{(--)} \in \{\mathfrak{C}, \mathfrak{TS}\}$ *then*
$$\mathfrak{O}^{(--)} P_k^{(+)} = P_{-k}^{(-)} \mathfrak{O}^{(--)}.$$

Proof. The relation are induced using the Riesz formula definition of the spectral eigenprojectors or direcly by Table 4.1. □

We now on focus on the 1D case, so $d = 1$, to give the following Theorem which is a concrete realization of what we have discussed in Subsection 2.3.2 about parallel transport.

Theorem 4.2.1 (Parallel transport)**.** *Let* $\{P_k\} \in \mathrm{M}_N(\mathbb{C})$ *be a family of projectors, which is at least* C^1 *in* k*. Then there exists a unique family of operators* $\{\mathcal{T}_k\} \in \mathrm{M}_N(\mathbb{C})$ *that solves the Cauchy problem*

$$\begin{cases} \partial_k \mathcal{T}_k = [\partial_k P_k, P_k]\mathcal{T}_k, \\ \mathcal{T}_0 = 1_{\mathbb{C}^N}. \end{cases}$$

The family of operators $\{\mathcal{T}_k\}$ *is at least as regular as* $\{P_k\}$ *as a function of* k*. Moreover,* $\mathcal{T}_k$ *satisfies the following properties:*

1. *Unitarity. Each* $\mathcal{T}_k$ *is unitary, i.e.* $(\mathcal{T}_k)^* = (\mathcal{T}_k)^{-1}$*.*
2. *Intertwining property. We have the relation*
$$P_k \mathcal{T}_k = \mathcal{T}_k P_0, \qquad \forall\, k \in \mathbb{R}.$$
3. *Telescopic property. If* P_k *is* 2π*-periodic in* k*, then*
$$\mathcal{T}_{k+2\pi n} = \mathcal{T}_k (\mathcal{T}_{2\pi})^n, \qquad \forall\, k \in \mathbb{R},\, n \in \mathbb{Z}.$$

We will call the family of unitaries $\mathcal{T}_k$ *the parallel transport operators associated with* $\{P_k\}$.

Proof. Existence, uniqueness, and regularity of the solution to the Cauchy problem follow directly from the standard theory of first-order linear differential equations. To prove unitarity, it is enough to note that the generator $[\partial_k P_k, P_k]$ is skew-adjoint

$$[\partial_k P_k, P_k]^* = -[\partial_k P_k, P_k],$$

so the solution $\mathcal{T}_k$ of the evolution equation is unitary for all k. Indeed, let us define the matrix-valued functions

$$U_k := \mathcal{T}_k^* \mathcal{T}_k, \qquad A_k := [\partial_k P_k, P_k].$$

We compute $\partial_k U_k$ using the product rule and original differential equation. We have

$$\begin{aligned} \partial_k U_k &= \partial_k (\mathcal{T}_k^* \mathcal{T}_k) \\ &= (\partial_k \mathcal{T}_k^*)\mathcal{T}_k + \mathcal{T}_k^*(\partial_k \mathcal{T}_k) \\ &= \mathcal{T}_k^* A_k^* \mathcal{T}_k + \mathcal{T}_k^* A_k \mathcal{T}_k \\ &= \mathcal{T}_k^*(A_k^* + A_k)\mathcal{T}_k. \end{aligned}$$

However, $A_k^* = -A_k$, so

$$\partial_k U_k = 0.$$

Hence, U_k is constant in k and since $\mathcal{T}_0 = 1_{\mathbb{C}^N}$, we have:

$$U_0 = \mathcal{T}_0^* \mathcal{T}_0 = 1_{\mathbb{C}^N},$$

so for all k we have

$$\mathcal{T}_k^* \mathcal{T}_k = 1_{\mathbb{C}^N} \quad \Rightarrow \quad \mathcal{T}_k^* = \mathcal{T}_k^{-1}.$$

The intertwining property clearly holds at $k = 0$, that is:

$$\mathcal{T}_0^* P_0 \mathcal{T}_0 = P_0.$$

To show that it holds for all $k \in \mathbb{R}$, we compute the derivative

$$\partial_k \left(\mathcal{T}_k^* P_k \mathcal{T}_k\right) = (\partial_k \mathcal{T}_k^*) P_k \mathcal{T}_k + \mathcal{T}_k^*(\partial_k P_k)\mathcal{T}_k + \mathcal{T}_k^* P_k(\partial_k \mathcal{T}_k).$$

Using the original equation and its adjoint, we obtain:

$$\partial_k(\mathcal{T}_k^* P_k \mathcal{T}_k) = \mathcal{T}_k^* \left(-[\partial_k P_k, P_k]P_k + \partial_k P_k + P_k[\partial_k P_k, P_k]\right)\mathcal{T}_k.$$

Now observe that the expression in parentheses vanishes identically due to the identity

$$P_k \partial_k P_k P_k = 0,$$

which follows from differentiating $P_k^2 = P_k$. Hence we have

$$\partial_k \left(\mathcal{T}_k^* P_k \mathcal{T}_k\right) = 0$$

which means that $\mathcal{T}_k^* P_k \mathcal{T}_k$ is constant in k. Since $\mathcal{T}_0 = 1_{\mathbb{C}^N}$, we conclude that

$$\mathcal{T}_k^* P_k \mathcal{T}_k = \mathcal{T}_0^* P_0 \mathcal{T}_0 = P_0,$$

so

$$P_k \mathcal{T}_k = \mathcal{T}_k P_0.$$

For the telescopic property, let us define $K_k := \mathcal{T}_{k+2\pi n}$ for fixed $n \in \mathbb{Z}$. Since P_k is 2π-periodic, the operator family $\{K_k\}$ solves the same original Cauchy problem

$$\partial_k K_k = [\partial_k P_k, P_k] K_k, \qquad K_0 = \mathcal{T}_{2\pi n}.$$

Now consider $\bar{K}_k := \mathcal{T}_k \mathcal{T}_{2\pi n}$ and

$$\partial_k \bar{K}_k = (\partial_k \mathcal{T}_k)\mathcal{T}_{2\pi n} = [\partial_k P_k, P_k]\mathcal{T}_k \mathcal{T}_{2\pi n} = [\partial_k P_k, P_k]\bar{K}_k,$$

and clearly $\bar{K}_0 = \mathcal{T}_0 \mathcal{T}_{2\pi n} = \mathcal{T}_{2\pi n} = K_0$. So K_k and $\bar{K}_k$ solve the same differential equation with the same initial condition. By uniqueness of solutions, we conclude that

$$\mathcal{T}_{k+2\pi n} = \mathcal{T}_k \mathcal{T}_{2\pi n}, \qquad \forall\ k \in \mathbb{R},\ n \in \mathbb{Z}.$$

Now, we proceed by induction. Fix $k = 0$ and $n = 1$ hence, trivially, $\mathcal{T}_{2\pi 1} = (\mathcal{T}_{2\pi})^1$. The inductive step assumes $\mathcal{T}_{2\pi n} = (\mathcal{T}_{2\pi})^n$ and, by the telescopic identity with $k = 2\pi$, we have

$$\mathcal{T}_{2\pi(n+1)} = \mathcal{T}_{2\pi}\mathcal{T}_{2\pi n} = \mathcal{T}_{2\pi}(\mathcal{T}_{2\pi})^n = (\mathcal{T}_{2\pi})^{n+1}.$$

Using the unitarity of $\{\mathcal{T}_k\}$, which implies that $(\mathcal{T}_{2\pi})^{-1} = \mathcal{T}_{-2\pi}$, we can deduce the same for negative n. The telescopic property follows. □

We note that the uniqueness of the solution of the Cauchy problem of Theorem 4.2.1 implies that the parallel transport operators associated with the families $P_k^{(-)}$ and $P_k^{(+)} = 1_{\mathbb{C}^N} - P_k^{(-)}$ coincide since they have the same generator of the linear differential equation

$$\left[\partial_k P_k^{(+)}, P_k^{(+)}\right] = \left[\partial_k (1_{\mathbb{C}^N} - P_k^{(-)}), 1_{\mathbb{C}^N} - P_k^{(-)}\right] = \left[\partial_k P_k^{(-)}, P_k^{(-)}\right]. \tag{4.19}$$

Moving on the same line of thought it is possible to show the following result.

Proposition 4.2.2 (Discrete symmetries and parallel transport)**.** *Let* $\{P_k^{(-)}, P_k^{(+)}\}$ *be a family of spectral eigenprojectors defined by* (4.17) *and* (4.18)*. Let* $\{\mathcal{T}_k\}$ *be the family of parallel transport unitaries associated with* $\{P_k^{(-)}\}$*. Then:*

1. *let* $\mathfrak{O}^{(+)} \in \left\{\mathfrak{O}^{(++)}, \mathfrak{O}^{(+-)}\right\}$*, i.e.* $\mathfrak{O}^{(+)} \in \{\mathfrak{S}, \mathfrak{T}\mathfrak{C}, \mathfrak{T}\mathfrak{C}\mathfrak{S}\}$ *then*

$$\mathfrak{O}^{(+)}\mathcal{T}_k = \mathcal{T}_k \mathfrak{O}^{(+)};$$

2. *let* $\mathfrak{O}^{(-)} \in \left\{\mathfrak{O}^{(-+)}, \mathfrak{O}^{(--)}\right\}$*, i.e.* $\mathfrak{O}^{(-)} \in \{\mathfrak{T}, \mathfrak{C}, \mathfrak{T}\mathfrak{S}, \mathfrak{C}\mathfrak{S}\}$ *then*

$$\mathfrak{O}^{(-)}\mathcal{T}_k = \mathcal{T}_{-k}\mathfrak{O}^{(-)}.$$

Proof. We take advantage once again of the uniqueness of the solution to the Cauchy problem. For the first case, let us define

$$U_k := \left(\mathfrak{D}^{(+)}\right)^{-1} \mathcal{T}_k \mathfrak{D}^{(+)};$$

differentiating, using Proposition 4.2.1 and (4.19), we get

$$\begin{aligned}\partial_k U_k &= \left(\mathfrak{D}^{(+)}\right)^{-1} (\partial_k \mathcal{T}_k) \mathfrak{D}^{(+)} = \left(\mathfrak{D}^{(+)}\right)^{-1} \left[\partial_k P_k^{(-)}, P_k^{(-)}\right] \mathcal{T}_k \mathfrak{D}^{(+)} = \\ &= \left[\left(\mathfrak{D}^{(+)}\right)^{-1} \partial_k P_k^{(-)} \mathfrak{D}^{(+)}, \left(\mathfrak{D}^{(+)}\right)^{-1} P_k^{(-)} \mathfrak{D}^{(+)}\right] \left(\mathfrak{D}^{(+)}\right)^{-1} \mathcal{T}_k \mathfrak{D}^{(+)} = \\ &= \left[\partial_k P_k^{(+/-)}, P_k^{(+/-)}\right] U_k;\end{aligned} \tag{4.20}$$

moreover, we have $U_0 = \left(\mathfrak{D}^{(+)}\right)^{-1} \mathcal{T}_0 \mathfrak{D}^{(+)} = 1_{\mathbb{C}^N}$. This means that U_k solves the same Cauchy problem of $\mathcal{T}_k$, hence

$$\left(\mathfrak{D}^{(+)}\right)^{-1} \mathcal{T}_k \mathfrak{D}^{(+)} = \mathcal{T}_k.$$

Similarly, let us define

$$U_k := \left(\mathfrak{D}^{(-)}\right)^{-1} \mathcal{T}_{-k} \mathfrak{D}^{(-)};$$

we have

$$\begin{aligned}\partial_k U_k &= \left(\mathfrak{D}^{(-)}\right)^{-1} (\partial_k \mathcal{T}_{-k}) \mathfrak{D}^{(-)} = \left(\mathfrak{D}^{(-)}\right)^{-1} \left[\partial_k P_{-k}^{(-)}, P_{-k}^{(-)}\right] \mathcal{T}_{-k} \mathfrak{D}^{(-)} = \\ &= \left[\left(\mathfrak{D}^{(-)}\right)^{-1} \partial_k P_{-k}^{(-)} \mathfrak{D}^{(-)}, \left(\mathfrak{D}^{(-)}\right)^{-1} P_{-k}^{(-)} \mathfrak{D}^{(-)}\right] \left(\mathfrak{D}^{(-)}\right)^{-1} \mathcal{T}_{-k} \mathfrak{D}^{(-)} = \\ &= \left[\partial_k P_k^{(+/-)}, P_k^{(+/-)}\right] U_k;\end{aligned} \tag{4.21}$$

moreover, we have $U_0 = \left(\mathfrak{D}^{(+)}\right)^{-1} \mathcal{T}_0 \mathfrak{D}^{(+)} = 1_{\mathbb{C}^N}$. Therefore

$$\left(\mathfrak{D}^{(-)}\right)^{-1} \mathcal{T}_{-k} \mathfrak{D}^{(-)} = \mathcal{T}_k.$$

□

Our goal is now to build up a symmetric Bloch basis. The following result ensure the existence and its proof show the explicit construction.

Theorem 4.2.2 (Existence of a symmetric Bloch basis)**.** *Let* $P_k^{(-)} \in \mathrm{M}_N(\mathbb{C})$ *be the rank-m eigenprojector defined by the Riesz formula in* (4.17)*. Let also* $\{v_1(0), ..., v_N(0)\} \in \mathbb{C}^N$ *be an orthonormal basis of eigenvectors of* H_0 *such that* $\{v_1(0), ..., v_m(0)\}$ *spans* $\mathrm{Range}\left(P_0^{(-)}\right)$ *while* $\{v_{m+1}(0), \dots, v_N(0)\}$ *spans* $\mathrm{Ker}\left(P_0^{(-)}\right) = \mathrm{Range}\left(1_{\mathbb{C}^N} - P_0^{(+)}\right)$*. Finally, denote by* $\{\mathcal{T}_k\}$ *the family of parallel transport unitaries associated with* $\left\{P_k^{(-)}\right\}$*. Then the family of vectors*

$$v_i(k) := \mathcal{T}_k e^{-ikX/2\pi} v_i(0), \qquad i \in \{1, \dots, N\},$$

where

$$X := -i \ln(\mathcal{T}_{2\pi}),$$

is a symmetric Bloch basis in the sense of Definition 4.2.2.

Proof. Let us first discuss the definition of X. We focus on the unitary matrix $\mathcal{T}_{2\pi} \in \mathrm{M}_N(\mathbb{C})$, called the holonomy unitary. By the spectral theorem it can be diagonalized by a unitary transformation V,

$$\mathcal{T}_{2\pi} = V^{-1} \begin{bmatrix} e^{i\varphi_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\varphi_N} \end{bmatrix} V, \qquad \varphi_i \in [0, 2\pi), \ \forall\, i \in \{1, \dots, N\}.$$

With this choice of the phases for the eigenvalues of $\mathcal{T}_{2\pi}$, we define $X \in \mathrm{M}_N(\mathbb{C})$ to be the self-adjoint matrix

$$X := -i \ln(\mathcal{T}_{2\pi}) = V^{-1} \begin{bmatrix} \varphi_1 & & 0 \\ & \ddots & \\ 0 & & \varphi_N \end{bmatrix} V,$$

so that $\mathcal{T}_{2\pi} = e^{iX}$.

Let us now show the spanning property. The vectors $\{v_i(k)\}_{i \in \{1,\dots,N\}}$ are clearly orthonormal, as they are obtained from the orthonormal basis of the eigenvectors of H_0 by means of unitary transformations. We note that the intertwining property of the parallel transport unitaries yields, in particular,

$$P^{(-)}_{2\pi} \mathcal{T}_{2\pi} = P^{(-)}_0 \mathcal{T}_{2\pi} = \mathcal{T}_{2\pi} P^{(-)}_0,$$

in view of the 2π-periodicity of the family of projections. Therefore, $\mathcal{T}_{2\pi}$ commutes with $P^{(-)}_0$, and as the matrix X is obtained from $\mathcal{T}_{2\pi}$ through functional calculus, so does X and, therefore, also $e^{-ikX/2\pi}$ for all $k \in \mathbb{R}$. It then follows that the vectors $e^{-ikX/2\pi} v_i(k)$ for $i \in \{1, \dots, m\}$ are orthonormal and in the range of $P^{(-)}_0$, and are then mapped orthonormally by $\mathcal{T}_k$ to the range of $P^{(-)}_k$ due to the intertwining property again.

The regularity property with respect to k is essentially inherited from the regularity of $P^{(-)}_k$. Indeed, from Theorem 4.2.1, $\mathcal{T}_k$ has the same regularity of $P^{(-)}_k$ with respect to k and the matrix $e^{-ikX/2\pi}$ depends analytically on k.

The 2π-periodicity follows from

$$\begin{aligned} v_i(k + 2\pi n) &= \mathcal{T}_{k+2\pi n} e^{-i(k+2\pi n)X/2\pi} v_i(0) = \mathcal{T}_k (\mathcal{T}_{2\pi})^n e^{-inX} e^{-ikX/2\pi} = \\ &= \mathcal{T}_k e^{-ikX/2\pi} v_i(0) = v_i(k), \end{aligned}$$

since $(\mathcal{T}_{2\pi})^n = e^{inX}$ by definition of X.

The last point is to prove that $\{v_i(k)\}_{i=1}^N$ satisfies the symmetry properties. We have to consider the four cases $\mathfrak{O}^{(++)}, \mathfrak{O}^{(+-)}, \mathfrak{O}^{(-+)}, \mathfrak{O}^{(--)}$ separately. Let us first consider the case of $\mathfrak{O}^{(++)}$. Since $\mathfrak{O}^{(++)}$ commutes with $\mathcal{T}_k$, in particular it commutes with $\mathcal{T}_{2\pi}$ and, by functional calculus, with $e^{-ikX/2\pi}$. Therefore

$$\begin{aligned} \mathfrak{O}^{(++)} v_i(k) &= \mathfrak{O}^{(++)} \mathcal{T}_k e^{-ikX/2\pi} v_i(0) = \mathcal{T}_k e^{-ikX/2\pi} \mathfrak{O}^{(++)} v_i(0) = \mathcal{T}_k e^{-ikX/2\pi} v_i(0) = \\ &= v_i(k). \end{aligned}$$

The case $\mathfrak{O}^{(+-)}$ is similar since it also commutes with $\mathcal{T}_k$ but we have

$$\begin{aligned} \mathfrak{O}^{(+-)} v_i(k) &= \mathfrak{O}^{(+-)} \mathcal{T}_k e^{-ikX/2\pi} v_i(0) = \mathcal{T}_k e^{-ikX/2\pi} \mathfrak{O}^{(+-)} v_i(0) = \\ &= \mathcal{T}_k e^{-ikX/2\pi} v_{N-i+1}(0) = v_{N-i+1}(k). \end{aligned}$$

In the case of $\mathfrak{O}^{(-+)}$, it commutes with $\mathcal{T}_k$ up to changing sign to k, hence, by functional calculus and anti-linearity, the same holds with $e^{-ikX/2\pi}$. Therefore

$$\begin{aligned}\mathfrak{O}^{(-+)}v_i(k) &= \mathfrak{O}^{(-+)}\mathcal{T}_k e^{-ikX/2\pi}v_i(0) = \mathcal{T}_{-k}e^{ikX/2\pi}\mathfrak{O}^{(-+)}v_i(0) = \mathcal{T}_{-k}e^{ikX/2\pi}v_i(0) = \\ &= v_i(-k).\end{aligned}$$

In a similar way

$$\begin{aligned}\mathfrak{O}^{(--)}v_i(k) &= \mathfrak{O}^{(--)}\mathcal{T}_k e^{-ikX/2\pi}v_i(0) = \mathcal{T}_{-k}e^{ikX/2\pi}\mathfrak{O}^{(--)}v_i(0) = \\ &= \mathcal{T}_{-k}e^{ikX/2\pi}v_{N-i+1}(0) = v_{N-i+1}(-k).\end{aligned}$$

In the case $\left(\mathfrak{O}^{(++)}\right)^2 = -1_{\mathbb{C}^{2m}}$ and/or $\left(\mathfrak{O}^{(-+)}\right)^2 = -1_{\mathbb{C}^{2m}}$ the same result follows using the linearity of $\mathcal{T}_k e^{-ikX/2\pi}$. □

4.3 The invariant

The topological invariant associated with the spectral eigenprojectors $P_k^{(-)}$ of the fiber Hamiltonians H_k will be extracted from any symmetric Bloch basis, whose existence is guaranteed by Theorem 4.2.2, starting form the abelian Berry phase of the Bloch basis.

4.3.1 Winding number for unitary matrices

We start with some results on the concept of winding number for unitary matrices. A first useful result we need is the following.

Proposition 4.3.1. *If $U : \mathbb{R} \to \mathrm{U}(N)$ is at least $\mathcal{C}^1$, then*

$$\partial_k \det(U_k) = \det(U_k)\,\mathrm{tr}\left(U_k^*\,\partial_k U_k\right).$$

Proof. Since U_k is unitary, we have $U_k U_k^* = 1_{\mathbb{C}^N}$. Using the first order expansion

$$\det(1_{\mathbb{C}^N} + H) = 1 + \mathrm{tr}(H) + \mathcal{O}(\|H\|^2),$$

we compute the derivative of $\det U_k$ as follows:

$$\begin{aligned}\partial_k \det(U_k) &= \lim_{h\to 0}\frac{\det(U_{k+h}) - \det(U_k)}{h} \\ &= \det(U_k)\lim_{h\to 0}\frac{\det\left(1_{\mathbb{C}^N} + U_k^*(U_{k+h} - U_k)\right) - 1}{h} \\ &= \det(U_k)\lim_{h\to 0}\frac{\mathrm{tr}\left(U_k^*(U_{k+h} - U_k)\right)}{h} \\ &= \det(U_k)\,\mathrm{tr}\left(U_k^*\,\partial_k U_k\right).\end{aligned}$$

□

The following result characterizes differential maps $f : S^1 \to S^1$ up to homotopy.

Proposition 4.3.2 (Winding number for a map $f : S^1 \to S^1$). *Let $f, g : S^1 \to S^1$ be two differentiable maps. The integral*

$$w(f) := \frac{1}{2\pi i}\int_{S^1} \frac{\partial_k f_k}{f_k} dk$$

is integer-valued and depends only on the homotopy class of f; it is called the winding number of f. The winding number of the product is the sum of the two winding numbers,

$$w(fg) = w(f) + w(g).$$

Moreover, let $\iota : S^1 \to S^1$ be the involution $k \mapsto -k$. For $f : S^1 \to S^1$, we then have

$$w(f \circ \iota) = -w(f).$$

Proof. The image of the map $f : S^1 \to S^1$ can be seen as a rectifiable and closed curve $\gamma \subset \mathbb{C}$. By the Cauchy integral formula,

$$\int_{S^1} \frac{\partial_k f_k}{f_k} dk = \int_\gamma \frac{dz}{z} = 2\pi i \operatorname{Ind}_\gamma(0),$$

where $\operatorname{Ind}_\gamma(0)$ computes the winding number of the curve γ around the origin in $\mathbb{C}$; this quantity is integer-valued.
Now, if $f^{(s)} : S^1 \to S^1$, with $s \in [0,1]$, is a smooth homotopy between two maps $f^{(0)}$ and $f^{(1)}$, then

$$\frac{d}{ds}\int_{S^1} \frac{\partial_k f_k^{(s)}}{f_k^{(s)}} dk = \int_{S^1} \frac{\partial_s \partial_k f_k^{(s)} \cdot f_k^{(s)} - \partial_k f_k^{(s)} \cdot \partial_s f_k^{(s)}}{(f_k^{(s)})^2} dk = \int_{S^1} \partial_k \left(\frac{\partial_s f_k^{(s)}}{f_k^{(s)}} \right) dk = 0,$$

and, therefore, $w(f)$ is constant along the homotopy class of f.
To show the additive property of the winding number, it suffices to compute

$$\begin{aligned} w(fg) &= \frac{1}{2\pi i}\int_{S^1} \frac{\partial_k (f_k g_k)}{f_k g_k} dk = \frac{1}{2\pi i}\int_{S^1} \left(\frac{\partial_k f_k}{f_k} + \frac{\partial_k g_k}{g_k} \right) dk = \\ &= \frac{1}{2\pi i}\int_{S^1} \frac{\partial_k f_k}{f_k} dk + \frac{1}{2\pi i}\int_{S^1} \frac{\partial_k g_k}{g_k} dk = w(f) + w(g). \end{aligned}$$

Finally, let $\iota : S^1 \to S^1$ be the involution $k \mapsto -k$. For $f : S^1 \to S^1$, we then have

$$(f \circ \iota)_k = f_{-k}, \quad \Rightarrow \quad \partial_k \left[(f \circ \iota)_k \right] = -(\partial_k f)_{-k} = -(\partial_k f) \circ \iota.$$

Therefore,

$$2\pi i\, w(f \circ \iota) = \int_0^{2\pi} \frac{\partial_k (f \circ \iota)_k}{(f \circ \iota)_k} dk = \int_0^{2\pi} \frac{-(\partial_k f)_{-k}}{f_{-k}} dk = -\int_0^{2\pi} \frac{(\partial_k f)_{-k}}{f_{-k}} dk.$$

Now, change variable $\kappa = -k$, so that when $k = 0$ we have $\kappa = 0$, and when $k = 2\pi$ we have $\kappa = -2\pi$, so

$$2\pi i\, w(f \circ \iota) = -\int_0^{2\pi} \frac{\partial_k f_{-k}}{f_{-k}} dk = \int_0^{-2\pi} \frac{\partial_\kappa f_\kappa}{f_\kappa} d\kappa = -\int_{-2\pi}^{0} \frac{\partial_\kappa f_\kappa}{f_\kappa} d\kappa.$$

We shift the bounds again by 2π-periodicity of f and its derivative to obtain

$$2\pi i\, w(f\circ\iota) = -\int_{-2\pi}^{0} \frac{\partial_\kappa f_\kappa}{f_\kappa} d\kappa = -\int_{0}^{2\pi} \frac{\partial_\kappa f_\kappa}{f_\kappa} d\kappa = -2\pi i\, w(f).$$

Hence,

$$w(f\circ\iota) = -w(f).$$

□

We can now state the main Theorem [23, 63].

Theorem 4.3.1 (Winding number of unitary-matrix-valued maps)**.** *Let $\pi_1(\mathrm{U}(N))$ denote the group of (smooth) homotopy classes of (smooth) maps $S^1 \to \mathrm{U}(N)$, where*

$$\mathrm{U}(N) := \{U \in \mathrm{M}_N(\mathbb{C}) \mid U^* = U^{-1}\},$$

endowed with point-wise multiplication of maps as the group operation. Then there is a group isomorphism $\pi_1(\mathrm{U}(N)) \simeq \mathbb{Z}$

$$\mathrm{wn} : [U : S^1 \to \mathrm{U}(N)] \mapsto \frac{1}{2\pi i}\int_{S^1} \mathrm{tr}\left(U_k^*\, \partial_k U_k\right) dk = \frac{1}{2\pi i}\int_{S^1} \frac{\partial_k \det(U_k)}{\det(U_k)} dk.$$

The integer $\mathrm{wn}([U])$ is called the winding number of the homotopy class $[U]$ of the map $U : S^1 \to \mathrm{U}(N)$.

Proof. Let $N \geq 1$. The group $\mathrm{SU}(N)$ has a natural action on $\mathbb{C}^N$ and also on the unit sphere within, namely the set

$$\{z \in \mathbb{C}^N : \|z\| = 1\},$$

which we can identify with S^{2N-1}. Then $\mathrm{SU}(N)$ acts transitively on S^{2N-1}. For a suitable point $z_0 \in S^{2N-1}$ the stabiliser of z_0 is $\mathrm{SU}(N-1)$. Therefore we may identify S^{2N-1} with the coset space $\mathrm{SU}(N)/\mathrm{SU}(N-1)$. This argument gives us the fibration

$$\begin{array}{ccc} \mathrm{SU}(N-1) & \longrightarrow & \mathrm{SU}(N) \\ & & \downarrow \\ & & S^{2N-1} \end{array}$$

from which we get a long exact sequence in homotopy

$$0 \longrightarrow \pi_2\left(S^{2N-1}\right) \longrightarrow \pi_1(\mathrm{SU}(N-1)) \longrightarrow \pi_1(\mathrm{SU}(N)) \longrightarrow \pi_1\left(S^{2N-1}\right) \longrightarrow 0.$$

Now, we want to show that $\pi_2(S^{2N-1}) = 0$. A sphere S^m has $\pi_k(S^m) = 0$ for all $k < m$; hence, if $2 < 2N-1 \Rightarrow N > 1$ we have $\pi_2(S^{2N-1}) = 0$ while if $N = 1$ we have $\pi_2(S^1) \simeq \pi_2(\mathbb{R}) = 0$. For $N = 1$ we have that $\mathrm{SU}(1) \simeq *$ and so it is simply connected. For $N > 1$ we have

$$\pi_1(\mathrm{SU}(N-1)) \simeq \pi_1(\mathrm{SU}(N))$$

since S^{2N-1} is simply connected. By induction we can show that $\pi_1(\mathrm{SU}(N)) = 0$: the inductive base is given by the $N = 1$ case and the inductive step is provided by the above isomorphism. At this point, we consider the short exact sequence

$$1 \longrightarrow \mathrm{SU}(N) \longrightarrow \mathrm{U}(N) \xrightarrow{\det} \mathrm{U}(1) \longrightarrow 1;$$

which gives us an isomorphism

$$\pi_1(\mathrm{U}(N)) \simeq \pi_1(\mathrm{U}(1))$$

since $\mathrm{SU}(N)$ is connected (this can be shown by explicit construction of the paths or by Lie group theory) and simply connected. Let now $U : S^1 \to \mathrm{U}(N)$ and let us consider

$$f := \det \circ U : S^1 \to \mathrm{U}(1).$$

We can compute the winding number

$$w(f) = \frac{1}{2\pi i}\int_{S^1} \frac{\partial_k \det(U_k)}{\det(U_k)} dk$$

which is integer valued. Therefore we have constructed the following chain of isomorphisms

$$\pi_1(\mathrm{U}(N)) \xrightarrow{\det_*} \pi_1(\mathrm{U}(1)) \xrightarrow{w} \mathbb{Z}.$$

□

4.3.2 Berry phase and its properties

We start with the definition of abelian Berry phase.

Definition 4.3.1 (Berry phase)**.** *Let $P_k \in \mathrm{M}_N(\mathbb{C})$ be a 2π-periodic family of rank-m projectors, which is at least C^1 in k and let $\{v_i(k)\}_{i=1}^N$ be a Bloch basis associated with P_k. The abelian Berry phase is defined as*

$$\mathcal{B}^{(N)}_{\{v_i(k)\}} := \frac{1}{2\pi i}\int_{S^1} \sum_{i=1}^{N} \langle v_i(k), \partial_k v_i(k)\rangle \, dk.$$

We are interested in computing the Berry phases of the Bloch basis constructed via parallel transport in Theorem 4.2.2.

Theorem 4.3.2 (Berry phase of a Bloch basis)**.** *Let*

$$v_i(k) := \mathcal{T}_k e^{-ikX/2\pi} v_i(0), \qquad i \in \{1, \dots, N\}$$

be the Bloch basis from Theorem 4.2.2 with

$$X := -i \ln \mathcal{T}_{2\pi}.$$

Then

$$\mathcal{B}^{(N)}_{\{v_i(k)\}} = \frac{1}{2\pi i}\int_{S^1} \sum_{i=1}^{N} \langle v_i(k), \partial_k v_i(k)\rangle dk = -\frac{1}{2\pi}\,\mathrm{tr}(X) \in \mathbb{Z}.$$

Proof. Let us define $W_k := \mathcal{T}_k e^{-ikX/2\pi}$ so that $v_i(k) = W_k v_i(0)$ for $i \in \{1, \dots, N\}$. We can then compute

$$\begin{aligned}\frac{1}{2\pi i}\int_{S^1}\sum_{i=1}^{N}\langle v_i(k), \partial_k v_i(k)\rangle dk &= \frac{1}{2\pi i}\int_{S^1}\sum_{i=1}^{N}\langle W_k v_i(0), \partial_k(W_k v_i(0))\rangle dk = \\ &= \frac{1}{2\pi i}\int_{S^1}\sum_{i=1}^{N}\langle v_i(0), W_k^*\partial_k W_k v_i(0)\rangle dk = \\ &= \frac{1}{2\pi i}\int_{S^1}\operatorname{tr}(W_k^*\partial_k W_k)dk,\end{aligned}$$

owing to the fact that $\{v_i(0)\}_{i=1}^N$ is an orthonormal basis of $\mathbb{C}^N$. Hence by Theorem 4.3.1 we have

$$\frac{1}{2\pi i}\int_{S^1}\sum_{i=1}^{N}\langle v_i(k), \partial_k v_i(k)\rangle dk = \operatorname{wn}([W]) \in \mathbb{Z}.$$

To prove the equality claimed in the statement, we compute

$$\begin{aligned}\partial_k W_k = \frac{d}{dk}\left(\mathcal{T}_k e^{-ikX/2\pi}\right) &= (\partial_k \mathcal{T}_k)e^{-ikX/2\pi} + \mathcal{T}_k\left(\frac{iX}{-2\pi}\right)e^{-ikX/2\pi} = \\ &= \left[\partial_k P_k^{(-)}, P_k^{(-)}\right] W_k + W_k\frac{X}{2\pi i};\end{aligned}$$

therefore,

$$W_k^*\partial_k W_k = W_k^*[\partial_k P_k^-, P_k^-]W_k + \frac{X}{2\pi i}.$$

Consequently,

$$\operatorname{tr}(W_k^*\partial_k W_k) = \operatorname{tr}(W_k^*[\partial_k P_k^-, P_k^-]W_k) + \operatorname{tr}\left(\frac{X}{2\pi i}\right).$$

Due to the invariance of the trace under unitary conjugation by W_k, we have that

$$\operatorname{tr}(W_k^*[\partial_k P_k^-, P_k^-]W_k) = \operatorname{tr}([\partial_k P_k^-, P_k^-]) = 0,$$

since commutators are traceless. Therefore, we conclude that

$$\operatorname{wn}([W]) = \frac{1}{2\pi i}\int_{S^1}\operatorname{tr}(W_k^*\partial_k W_k)\,dk = \frac{1}{(2\pi i)^2}\operatorname{tr}(X)\int_{S^1}dk = \frac{2\pi}{(2\pi i)^2}\operatorname{tr}(X).$$

□

Since $\mathcal{T}_{2\pi} = e^{iX}$ and, due to the above Theorem 4.3.2, we have that $\operatorname{tr}(X) = -2\pi n$ for $n = \operatorname{wn}([W]) \in \mathbb{Z}$, it must be that

$$\det(\mathcal{T}_{2\pi}) = e^{i\operatorname{tr}(X)} = e^{2\pi i n} = 1.$$

since the relation $\det(\mathcal{T}_{2\pi}) = e^{\operatorname{tr}(\ln(\mathcal{T}_{2\pi}))} = e^{i\operatorname{tr}(X)}$ holds. Therefore, while the parallel transport unitaries $\mathcal{T}_k$ are not 2π-periodic in k in general, the determinant map $\Phi : k \mapsto \det(\mathcal{T}_k)$ is 2π-periodic and so has a well-defined winding number in view of Proposition 4.3.2, which can be alternatively used to compute, according to Theorem 4.3.1, the Berry phase of the Bloch basis $\{v_i(k)\}_{i=1}^N$.
We are now interested in considering a change of Bloch basis. We first give the following Definition.

Definition 4.3.2 (Bloch gauge). *Let $\{v_i(k)\}_{i=1}^N$ be a Bloch basis and let $\{u_i(k)\}_{i=1}^N$ be another Bloch basis. The change of basis matrix B_k such that $u_i(k) = B_k v_i(k)$ for all $i \in \{1, ..., N\}$ is called Bloch gauge.*

By definition, a Bloch basis $\{u_i(k)\}_{i=1}^N$ associated to $P_k^{(-)}$ has the first m vectors in Range $\left(P_k^{(-)}\right)$ and the last $N-m$ in Range $\left(P_k^{(+)}\right)$. Therefore, the Bloch gauge B_k which maps the Bloch basis $\{v_i(k)\}_{i=1}^N$ to $\{u_i(k)\}_{i=1}^N$ has a block-diagonal form in the decomposition $\mathbb{C}^N = \text{Range}\left(P_k^{(-)}\right) \oplus \text{Range}\left(P_k^{(+)}\right)$, i.e.

$$B_k = \begin{bmatrix} B_k^{(-)} & 0 \\ 0 & B_k^{(+)} \end{bmatrix};$$

where $B_k^{(\pm)} = P_k^{(\pm)} B_k P_k^{(\pm)}$, seen as a linear operator on Range $\left(P_k^{(\pm)}\right)$. In particular,

$$u_i(k) = \begin{cases} B_k^{(-)} v_i(k) & \text{if} \quad i \in \{1, \dots, m\}, \\ B_k^{(+)} v_i(k) & \text{if} \quad i \in \{m+1, \dots, N\}. \end{cases}$$

The next result shows that when a change of Bloch basis is performed the Berry phase changes by an integer.

Theorem 4.3.3 (Bloch basis change and Berry phase). *Let $\{v_i(k)\}_{i=1}^N$ be a Bloch basis and let $\{u_i(k)\}_{i=1}^N$ be another Bloch basis. Let B_k the change of basis matrix. Then B_k is unitary, 2π-periodic and as regular in k as the Bloch bases. Moreover,*

$$\mathcal{B}^{(N)}_{\{u_i(k)\}} - \mathcal{B}^{(N)}_{\{v_i(k)\}} = \text{wn}([B]) \in \mathbb{Z},$$

where wn($[B]$) *denotes the winding number of the map*

$$B : S^1 \to U(N), \quad k \mapsto B_k.$$

Proof. For each k, the matrix B_k is unitary, since it maps one orthonormal basis into another. Moreover, in the 2π-periodic and regular orthonormal basis $\{v_i(k)\}_{i=1}^N$ for $\mathbb{C}^N$, the Bloch gauge B_k has entries

$$\langle v_j(k), B_k v_i(k)\rangle = \langle v_j(k), u_i(k)\rangle, \qquad i, j \in \{1, \dots, N\},$$

and, therefore, it is itself 2π-periodic and regular. We have

$$\begin{aligned}
\mathcal{B}^{(N)}_{\{u_i(k)\}} &= \frac{1}{2\pi i}\int_{S^1} \sum_{i=1}^N \langle B_k v_i(k), \partial_k (B_k v_i(k))\rangle dk = \\
&= \frac{1}{2\pi i}\int_{S^1} \sum_{i=1}^N \langle v_i(k), \partial_k v_i(k)\rangle dk + \frac{1}{2\pi i}\int_{S^1} \sum_{i=1}^N \langle v_i(k), B_k^* \partial_k B_k v_i(k)\rangle dk = \\
&= \frac{1}{2\pi i}\int_{S^1} \sum_{i=1}^N \langle v_i(k), \partial_k v_i(k)\rangle dk + \frac{1}{2\pi i}\int_{S^1} \text{tr}(B_k^* \partial_k B_k) dk = \\
&= \mathcal{B}^{(N)}_{\{v_i(k)\}} + \text{wn}([B]).
\end{aligned}$$

□

In combination with the previous result, we can conclude that Berry phases are always integer-valued.

4.3.3 The $\mathbb{Z}_2$ invariant

The point is now to understand what values the term $\mathrm{wn}([B])$ can take. In the case of only chiral or particle hole symmetry we have $\mathrm{wn}([B]) \in 2\mathbb{Z}$ and so taking the mod-2 remainder of the Berry phase we get a $\mathbb{Z}_2$ object that can be shown to be a topological invariant. In our case we have all the possible discrete symmetries and what we get is that $\mathrm{wn}([B]) \in 2\mathbb{Z}$ as stated from the following result.

Theorem 4.3.4 (Gauge transformation of Berry phase). *Let $\{v_i(k)\}_{i=1}^N$ be a symmetric Bloch basis and let $\{u_i(k)\}_{i=1}^N$ be another symmetric Bloch basis. Let B_k the Bloch gauge then*

$$\mathrm{wn}([B]) \in 2\mathbb{Z}.$$

Proof. We will use all the symmetries at our disposal to understand what values $\mathrm{wn}([B])$ can take. First of all we have

$$\begin{aligned}\mathrm{wn}([B]) = w(\det(B_k)) = w\left(\det\left(B_k^{(+)}\right)\det\left(B_k^{(-)}\right)\right) =\\ = w\left(\det\left(B_k^{(+)}\right)\right) + w\left(\det\left(B_k^{(-)}\right)\right).\end{aligned}$$

Let us start with the operator $\mathfrak{O}^{(++)}$; we have

$$\mathfrak{O}^{(++)}v_i(k) = v_i(k), \quad \text{and} \quad \mathfrak{O}^{(++)}u_i(k) = u_i(k), \qquad \forall\ i \in \{1, ..., 2m\},$$

and

$$\mathfrak{O}^{(++)}P_k^{(+/-)} = P_k^{(+/-)}\mathfrak{O}^{(++)}.$$

We have

$$\begin{aligned}B_k^{(-)}\mathfrak{O}^{(++)}v_i(k) = B_k^{(-)}v_i(k) = u_i(k) = \mathfrak{O}^{(++)}u_i(k) = \mathfrak{O}^{(++)}B_k^{(-)}v_i(k);\\ B_k^{(+)}\mathfrak{O}^{(++)}v_i(k) = B_k^{(+)}v_i(k) = u_i(k) = \mathfrak{O}^{(++)}u_i(k) = \mathfrak{O}^{(++)}B_k^{(+)}v_i(k);\end{aligned}$$

the first one for all $i \in \{1, ..., m\}$ and the second one for $i \in \{m+1, ..., N\}$. Therefore

$$\begin{aligned}B_k^{(-)}P_k^{(-)}\mathfrak{O}^{(++)} = B_k^{(-)}\mathfrak{O}^{(++)}P_k^{(-)} = \mathfrak{O}^{(++)}B_k^{(-)}P_k^{(-)};\\ B_k^{(+)}P_k^{(+)}\mathfrak{O}^{(++)} = B_k^{(+)}\mathfrak{O}^{(++)}P_k^{(+)} = \mathfrak{O}^{(++)}B_k^{(+)}P_k^{(+)};\end{aligned}$$

from which we deduce that

$$\begin{aligned}P_k^{(-)}B_kP_k^{(-)}\mathfrak{O}^{(++)} = \mathfrak{O}^{(++)}P_k^{(-)}B_kP_k^{(-)} \quad &\Rightarrow \quad B_k^{(-)}\mathfrak{O}^{(++)} = \mathfrak{O}^{(++)}B_k^{(-)};\\ P_k^{(+)}B_kP_k^{(+)}\mathfrak{O}^{(++)} = \mathfrak{O}^{(++)}P_k^{(+)}B_kP_k^{(+)} \quad &\Rightarrow \quad B_k^{(+)}\mathfrak{O}^{(++)} = \mathfrak{O}^{(++)}B_k^{(+)}.\end{aligned}$$

The above relations, since $\mathfrak{O}^{(++)}$ is linear, do not provide non-trivial information on the winding number of $B^{(\pm)}$.

Let us now consider the case of $\mathfrak{O}^{(+-)}$; in this case

$$\mathfrak{O}^{(+-)}\,v_i(k) = v_{N-i+1}(k), \quad \text{and} \quad \mathfrak{O}^{(+-)}\,u_i(k) = u_{N-i+1}(k), \qquad \forall\ i \in \{1, ..., m\},$$

and

$$\mathfrak{O}^{(+-)}P_k^{(+)} = P_k^{(-)}\mathfrak{O}^{(+-)}.$$

We have

$$B_k^{(+)}\mathfrak{O}^{(+-)}v_i(k) = B_k^{(+)}v_{N-i+1}(k) = u_{N-i+1}(k) = \mathfrak{O}^{(+-)}u_i(k) = \mathfrak{O}^{(+-)}B_k^{(-)}v_i(k)$$

for all $i \in \{1, ..., m\}$. Therefore

$$B_k^{(+)}P_k^{(+)}\mathfrak{O}^{(+-)} = B_k^{(+)}\mathfrak{O}^{(+-)}P_k^{(-)} = \mathfrak{O}^{(+-)}B_k^{(-)}P_k^{(-)};$$

from which we deduce that

$$P_k^{(+)}B_kP_k^{(+)}\mathfrak{O}^{(+-)} = \mathfrak{O}^{(+-)}P_k^{(-)}B_kP_k^{(-)} \quad \Rightarrow \quad B_k^{(+)}\mathfrak{O}^{(+-)} = \mathfrak{O}^{(+-)}B_k^{(-)}.$$

The above relation, since $\mathfrak{O}^{(+-)}$ is linear, gives

$$\det\left(B_k^{(+)}\right) = \det\left(B_k^{(-)}\right).$$

Therefore we have

$$\mathrm{wn}([B]) = 2w\left(\det\left(B_k^{(+)}\right)\right) \in 2\mathbb{Z}.$$

It is now the turn of $\mathfrak{O}^{(--)}$; we have

$$\mathfrak{O}^{(+-)}v_i(k) = v_{N-i+1}(-k), \quad \text{and} \quad \mathfrak{O}^{(+-)}u_i(k) = u_{N-i+1}(-k), \qquad \forall\, i \in \{1, ..., m\},$$

and

$$\mathfrak{O}^{(--)}P_k^{(+)} = P_{-k}^{(-)}\mathfrak{O}^{(--)}.$$

We have

$$B_{-k}^{(+)}\mathfrak{O}^{(--)}v_i(k) = B_{-k}^{(+)}v_{N-i+1}(-k) = u_{N-i+1}(-k) = \mathfrak{O}^{(--)}u_i(k) = \mathfrak{O}^{(--)}B_k^{(-)}v_i(k)$$

for all $i \in \{1, ..., m\}$. Therefore

$$B_{-k}^{(+)}P_{-k}^{(+)}\mathfrak{O}^{(--)} = B_{-k}^{(+)}\mathfrak{O}^{(--)}P_k^{(-)} = \mathfrak{O}^{(--)}B_k^{(-)}P_k^{(-)};$$

from which we deduce that

$$P_{-k}^{(+)}B_{-k}P_{-k}^{(+)}\mathfrak{O}^{(--)} = \mathfrak{O}^{(--)}P_k^{(-)}B_kP_k^{(-)} \quad \Rightarrow \quad B_{-k}^{(+)}\mathfrak{O}^{(--)} = \mathfrak{O}^{(--)}B_k^{(-)}.$$

The above relation, since $\mathfrak{O}^{(+-)}$ is anti-linear, gives

$$\det\left(B_{-k}^{(+)}\right) = \left(\det\left(B_k^{(-)}\right)\right)^* = \det\left(\left(B_k^{(-)}\right)^*\right) = \det\left(\left(B_k^{(-)}\right)^{-1}\right).$$

Therefore we have again

$$\mathrm{wn}([B]) = 2w\left(\det\left(B_k^{(+)}\right)\right) \in 2\mathbb{Z},$$

since

$$w\left(\det\left(B_k^{(+)}\right)\right) = -w\left(\det\left(B_{-k}^{(+)}\right)\right) = -w\left(\det\left(\left(B_k^{(-)}\right)^{-1}\right)\right) = w\left(\det\left(B_k^{(-)}\right)\right).$$

Last but not least, the operator $\mathfrak{O}^{(-+)}$; we have

$$\mathfrak{O}^{(-+)} v_i(k) = v_i(-k), \quad \text{and} \quad \mathfrak{O}^{(-+)} u_i(k) = u_i(-k), \qquad \forall\, i \in \{1, ..., 2m\},$$

and

$$\mathfrak{O}^{(-+)} P_k^{(+/-)} = P_{-k}^{(+/-)} \mathfrak{O}^{(-+)}.$$

We have

$$\begin{aligned} B_{-k}^{(-)} \mathfrak{O}^{(-+)} v_i(k) = B_{-k}^{(-)} v_i(-k) = u_i(-k) = \mathfrak{O}^{(-+)} u_i(k) = \mathfrak{O}^{(-+)} B_k^{(-)} v_i(k); \\ B_{-k}^{(+)} \mathfrak{O}^{(-+)} v_i(k) = B_{-k}^{(+)} v_i(-k) = u_i(-k) = \mathfrak{O}^{(-+)} u_i(k) = \mathfrak{O}^{(-+)} B_k^{(+)} v_i(k); \end{aligned}$$

the first one for all $i \in \{1, ..., m\}$ and the second one for $i \in \{m+1, ..., N\}$. Therefore

$$\begin{aligned} B_{-k}^{(-)} P_{-k}^{(-)} \mathfrak{O}^{(-+)} = B_{-k}^{(-)} \mathfrak{O}^{(-+)} P_k^{(-)} = \mathfrak{O}^{(-+)} B_k^{(-)} P_k^{(-)}; \\ B_{-k}^{(+)} P_{-k}^{(+)} \mathfrak{O}^{(-+)} = B_{-k}^{(+)} \mathfrak{O}^{(-+)} P_k^{(+)} = \mathfrak{O}^{(-+)} B_k^{(+)} P_k^{(+)}; \end{aligned}$$

from which we deduce that

$$\begin{aligned} P_{-k}^{(-)} B_{-k} P_{-k}^{(-)} \mathfrak{O}^{(-+)} = \mathfrak{O}^{(-+)} P_k^{(-)} B_k P_k^{(-)} \quad \Rightarrow \quad B_{-k}^{(-)} \mathfrak{O}^{(-+)} = \mathfrak{O}^{(-+)} B_k^{(-)}; \\ P_{-k}^{(+)} B_{-k} P_{-k}^{(+)} \mathfrak{O}^{(-+)} = \mathfrak{O}^{(-+)} P_k^{(+)} B_k P_k^{(+)} \quad \Rightarrow \quad B_{-k}^{(+)} \mathfrak{O}^{(-+)} = \mathfrak{O}^{(-+)} B_k^{(+)}. \end{aligned}$$

The above relations, since $\mathfrak{O}^{(-+)}$ is anti-linear, give

$$\begin{aligned} \det\left(B_{-k}^{(+)}\right) = \left(\det\left(B_k^{(+)}\right)\right)^*; \\ \det\left(B_{-k}^{(-)}\right) = \left(\det\left(B_k^{(-)}\right)\right)^*. \end{aligned}$$

Therefore we have the set of relations:

$$\begin{aligned} \det\left(B_{-k}^{(+)}\right) &= \left(\det\left(B_k^{(-)}\right)\right)^*; \\ \det\left(B_k^{(+)}\right) &= \det\left(B_k^{(-)}\right); \\ \det\left(B_{-k}^{(+)}\right) &= \left(\det\left(B_k^{(+)}\right)\right)^*; \\ \det\left(B_{-k}^{(-)}\right) &= \left(\det\left(B_k^{(-)}\right)\right)^*; \end{aligned}$$

however, some of them are redundant and the above four relations are equivalent to the pair:

$$\begin{aligned} \det\left(B_k^{(+)}\right) &= \det\left(B_k^{(-)}\right); \\ \det\left(B_{-k}^{(+)}\right) &= \left(\det\left(B_k^{(+)}\right)\right)^*. \end{aligned}$$

The second relation indicates that the map $k \mapsto \det\left(B_k^{(+)}\right)$ is such that the image of the map is symmetric under reflection about the real axis in the complex plane. If we write

$$\det\left(B_k^{(+)}\right) = e^{i\theta(k)},$$

then we have

$$\left(\det\left(B_k^{(+)}\right)\right)^* = e^{-i\theta(k)} = \det\left(B_{-k}^{(+)}\right) = e^{i\theta(-k)} \quad \Rightarrow \quad \theta(-k) = -\theta(k) + 2\pi n_2,$$

and from the 2π-periodicity of $B_k^{(+)}$ we get

$$\det\left(B_{k+2\pi}^{(+)}\right) = e^{i\theta(k+2\pi)} = \det\left(B_k^{(+)}\right) = e^{i\theta(k)} \quad \Rightarrow \quad \theta(k+2\pi) = \theta(k) + 2\pi n_1,$$

with $n_1, n_2 \in \mathbb{Z}$. Moreover

$$\begin{aligned} w\left(\det\left(B_k^{(+)}\right)\right) &= \frac{1}{2\pi i}\int_{S^1} \frac{\partial_k \det\left(B_k^{(+)}\right)}{\det\left(B_k^{(+)}\right)} dk = \frac{1}{2\pi i}\int_{S^1} \frac{i(\partial_k\theta(k))e^{i\theta(k)}}{e^{i\theta(k)}} dk = \\ &= \frac{1}{2\pi}\int_{S^1} \partial_k\theta(k) dk = \frac{\theta(2\pi)-\theta(0)}{2\pi} = n_1 \in \mathbb{Z}. \end{aligned}$$

Therefore $\mathrm{wn}([B]) \in 2\mathbb{Z}$. The same result follows in the case $\left(\mathfrak{O}^{(++)}\right)^2 = -1_{\mathbb{C}^{2m}}$ and/or $\left(\mathfrak{O}^{(-+)}\right)^2 = -1_{\mathbb{C}^{2m}}$ by using the linearity of $B_k^{(+)}$ and of $B_k^{(-)}$. □

We can then conclude the following result.

Corollary 4.3.1 (Gauge invariance of mod 2 Berry phase)**.** *The mod 2 equivalence class*

$$\left[\frac{1}{2\pi i}\int_{S^1} dk \sum_{i=1}^{2m} \langle v_i(k), \partial_k v_i(k)\rangle\right] = \left[\frac{1}{\pi i}\int_{S^1} dk \sum_{i=1}^{m} \langle v_i(k), \partial_k v_i(k)\rangle\right] \in \mathbb{Z}_2,$$

does not depend on the choice of a symmetric Bloch basis $\{v_i(k)\}_{i=1}^N$ associated with $P_k^{(-)}$.

Proof. The independence of the mod 2 equivalence of the Berry phase from the Bloch basis is a direct consequence of the above Theorem. Hence the mod 2 equivalence of the Berry phase is a gauge invariant. The rewriting follows from using the actions of operators $\mathfrak{O}^{(+-)}$ and $\mathfrak{O}^{(--)}$ in Definition 4.2.2 and manipulating the scalar product in the definition of the Berry phase. □

We therefore can give the following Definition.

Definition 4.3.3 (The invariant)**.** *Let H be a finite range Hamiltonian on $\ell^2(\mathbb{Z})\otimes\mathbb{C}^N$ modeling a translation invariant insulator with time reversal, particle hole and chiral symmetries. Then the quantity*

$$\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H) := \frac{1}{\pi i}\int_{S^1} dk \sum_{i=1}^{m} \langle v_i(k), \partial_k v_i(k)\rangle \quad \mathrm{mod}\ 2 \in \mathbb{Z}_2,$$

where is $\{v_i(k)\}_{i=1,\dots,m}$ is any symmetric Bloch basis, is a gauge invariant of the spectral projection of H onto the negative energy eigenspace.

The last point is the topological invariance of the quantity $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$.

Theorem 4.3.5 (Topological invariance of $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$)**.** *Assume that* $H(t)$*,* $t \in [0,1]$*, is a continuous family of bounded Hamiltonian operators on* $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$ *describing topological insulators in symmetry classes* BDI, DIII, CII, CI. *Assume moreover that the spectral gap of* $H(t)$ *remains open for all* $t \in [0,1]$*. Then*

$$\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H(0)) = \mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H(1)).$$

Proof. We follow the proof of [54]. The homotopy $t \mapsto H(t)$ induces a corresponding homotopy

$$t \mapsto P_k^{(-)}(t)$$

between the spectral eigenprojectors of the fiber operators $H_k(t)$, which remains continuous due to the fact that the spectral gap remains open. Since both $k \in S^1$ and $t \in [0,1]$ range over compact sets, by uniform continuity there exists $\delta > 0$ such that

$$\sup_{k \in S^1} \left\| P_k^{(-)}(t) - P_k^{(-)}(s) \right\| < 1 \qquad \text{whenever} \quad |t-s| < \delta.$$

Thanks to Proposition 2.2.4, we have a Kato-Nagy unitary operator $U_k(t,s)$ such that

$$P_k^{(-)}(t) U_k(t,s) = U_k(t,s) P_k^{(-)}(s).$$

From the explicit expression for the Kato-Nagy unitary, it follows that $U_k(t,s)$ is as regular in k as the eigenprojectors, jointly continuous in t and s and commutes with all the discrete symmetry operators up to a possible composition with the involution $k \mapsto -k$.

Without loss of generality, fix $s = 0$, and let $\{v_i(k;0)\}_{i=1}^N$ be a symmetric Bloch basis associated with $P_k^{(-)}(0)$. By the properties of the Kato-Nagy unitary mentioned above, the collection of vectors

$$v_i(k;t) := U_k(t,0) v_i(k;0), \qquad i \in \{1, \dots, N\},$$

defines a symmetric Bloch basis associated with $P_k^{(-)}(t)$ as long as $|t| < \delta$. By computations similar to those of Theorem 4.3.4, it follows that the Berry phase of the Bloch basis $\{\{v_i(k;t)\}_{i=1}^N$ has the same parity as that of the Bloch basis $\{\{v_i(k;0)\}_{i=1}^N$, which defines the invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H(0))$. This implies that $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H(t))$ is constant on the interval $[0, \delta)$.

By partitioning the interval $[0,1]$ into finitely many subintervals of length less than $\delta/2$ and iterating the above argument on each subinterval, we conclude that the invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H(t))$ remains constant on the whole interval $[0,1]$. Therefore, it is a homotopic invariant. □

4.3.4 A refinement for symmetry classes with quaternionic structure

Here we want to refine the $\mathbb{Z}_2$ topological invariant. We are going to make explicit the restrictions that the presence of an anti-linear operator with square equal to minus the identity can give. What we will obtain is an equivariant quaternionic topological gauge invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$. Let us start with the following Definition.

Definition 4.3.4 (Complex and quaternionic structure)**.** *Let V be a vector real space; if there exist a linear map $J_c : V \to V$ such that $J_c^2 = -1_V$ then V is said to be endowed by a complex structure. In an analogue way, let V be is a complex vector space; if there exist an anti-linear map $J_h : V \to V$ such that $J_h^2 = -1_V$ then V is said to be endowed by a quaternionic structure.*

The first result we need is the following.

Proposition 4.3.3 (Quaternionic structure and unitary matrices)**.** *Let us consider the Hilbert space $\mathbb{C}^{2m}$ endowed with a quaternionic structure, i.e an anti-linear operator O such that $O^2 = -1_{\mathbb{C}^{2m}}$. Let us assume that O is anti-unitary. Then, a unitary matrix U commuting with O has spectrum symmetric under complex conjugation with even multiplicities and its determinant is forced to be* $\det(U) = 1$.

Proof. Let $U \in \mathrm{U}(2m)$ be a unitary matrix acting on a complex Hilbert space $\mathbb{C}^{2m}$, and let O be an anti-unitary operator such that

$$O^2 = -1_{\mathbb{C}^{2m}} \qquad \text{and} \qquad OUO^{-1} = U.$$

Since U is unitary, all its eigenvalues lie on the complex unit circle. Let $\lambda \in \mathbb{C}$ be an eigenvalue of U with eigenvector $v \in \mathbb{C}^{2m}$, so that

$$U(v) = \lambda v.$$

Applying O to both sides and using anti-linearity:

$$O(U(v)) = O(\lambda v) = \lambda^* O(v).$$

Since O commutes with U, we have that

$$U(O(v)) = O(U(v)) = \lambda^* O(v),$$

so $O(v)$ is an eigenvector of U with eigenvalue λ^*. We claim that v and $O(v)$ are linearly independent. Suppose, by contradiction, that $O(v) = \alpha v$ for some $\alpha \in \mathbb{C}$. Then:

$$O(O(v)) = O(\alpha v) = \alpha^* O(v) = \alpha^* \alpha v = |\alpha|^2 v.$$

Since $O^2 = -1_{\mathbb{C}^{2m}}$, this implies $|\alpha|^2 v = -v$, hence $|\alpha|^2 = -1$, which is impossible in $\mathbb{C}$. Therefore, v and $O(v)$ are linearly independent. It follows that the eigenvalues of U come in complex conjugate pairs with equal multiplicity. Now, let the eigenvalues of U be $\lambda_1, \lambda_1^*, \dots, \lambda_m, \lambda_m^*$. Then we have

$$\det(U) = \prod_{j=1}^{m} \lambda_j \lambda_j^* = \prod_{j=1}^{m} |\lambda_j|^2 = 1,$$

since $|\lambda_j| = 1$ for all j because U is unitary. □

We note, by contrast, that if $O^2 = +1_{\mathbb{C}^{2m}}$ no such constraint holds: eigenvalues may occur without pairing, and $\det(U)$ can take any value in $\mathrm{U}(1)$. Moreover, if O is linear and $O^2 = -1_{\mathbb{C}^{2m}}$ we get the constraint $\alpha^2 = -1$ which implies that v and $O(v)$ are linearly dependent and the conclusion does not hold.
Let us consider now the matrix $W_k = \mathcal{T}_k e^{-ikX/2\pi}$. We have the following fundamental result.

Theorem 4.3.6 (Berry phase and quaternionic structure)**.** *Let us consider the Hilbert space* $\mathbb{C}^{2m}$ *endowed with a quaternionic structure due to an anti-linear operator* $\mathfrak{O}^{(-)}$ *such that it squares to* $-1_{\mathbb{C}^{2m}}$*. Then* $\mathrm{wn}([W]) \in 2\mathbb{Z}$*.*

Proof. We first prove that W_k is unitary and 2π-periodic. Indeed

$$W_k^* W_k = \left(\mathcal{T}_k e^{-ikX/2\pi}\right)^* \mathcal{T}_k e^{-ikX/2\pi} = \left(e^{-ikX/2\pi}\right)^* \mathcal{T}_k^* \mathcal{T}_k e^{-ikX/2\pi} = 1_{\mathbb{C}^{2m}},$$

since $X = -i\ln(\mathcal{T}_{2\pi})$ is self adjoint. Moreover we have

$$W_{k+2\pi} = \mathcal{T}_{k+2\pi} e^{-i(k+2\pi)X/2\pi} = \mathcal{T}_k \mathcal{T}_{2\pi} e^{-ikX/2\pi} e^{-iX} = \mathcal{T}_k \mathcal{T}_{2\pi} \left(e^{iX}\right)^{-1} e^{-ikX/2\pi} = W_k.$$

Therefore, W_π commutes with the time reversal and/or particle hole symmetry operator due to Proposition 4.2.2 and the 2π-periodicity

$$\mathfrak{O}^{(-)} W_\pi = W_{-\pi} \mathfrak{O}^{(-)} = W_\pi \mathfrak{O}^{(-)};$$

by Proposition 4.3.3 we conclude that $\det(W_\pi) = \det(W_{-\pi}) = 1$. Let us now compute the winding number

$$\mathrm{wn}([W]) = w(\det(W_k)) = \frac{1}{2\pi i} \int_{S^1} \frac{\partial_k \det(W_k)}{\det(W_k)} dk.$$

Since W_k is unitary we can write

$$\det(W_k) = e^{i\theta(k)},$$

for some smooth real-valued function $\theta(k)$; hence, by 2π-periodicity of W_k we have

$$\theta(k + 2\pi) = \theta(k) + 2\pi n_1, \qquad n_1 \in \mathbb{Z}.$$

Since $\mathfrak{O}^{(-)}$ is anti-unitary we have

$$\det(W_{-k}) = \det\left(\mathfrak{O}^{(-)} W_k \left(\mathfrak{O}^{(-)}\right)^{-1}\right) = (\det(W_k))^* = e^{-i\theta(k)},$$

so that

$$\theta(-k) = -\theta(k) + 2\pi n_2, \qquad n_2 \in \mathbb{Z}.$$

Differentiating both sides gives:

$$-\theta'(-k) = -\theta'(k),$$

which implies that $\theta'(k)$ is an even function.
Now,

$$w(\det(W_k)) = \frac{1}{2\pi} \int_0^{2\pi} \theta'(k) dk = \frac{1}{2\pi}(\theta(2\pi) - \theta(0)) = n_1.$$

Since $\det(W_\pi) = e^{i\theta(\pi)} = 1$, it must be

$$\theta(\pi) = 2\pi n_\pi, \qquad n_\pi \in \mathbb{Z};$$

and so, from pseudo-oddness of $\theta(k)$, we have

$$\theta(-\pi) = -\theta(\pi) + 2\pi n_2 = 2\pi(n_2 - n_\pi).$$

Moreover,

$$\theta(-\pi) = -\theta(-\pi + 2\pi) + 2\pi n_2 = -\theta(-\pi) + 2\pi(n_2 - n_1) \quad \Rightarrow \quad \theta(-\pi) = \pi(n_2 - n_1),$$

so

$$n_1 = -\frac{\theta(-\pi)}{\pi} + n_2 = -\frac{2\pi(n_2 - n_\pi)}{\pi} + n_2 = -n_2 + 2n_\pi.$$

However, since $\det(W_0) = \det(\mathcal{T}_0) = 1$ by definition, we have $e^{i\theta(0)} = 1$ so $\theta(0) = 2\pi n_0$ with $n_0 \in \mathbb{Z}$ but, from the pseudo-oddness, the relation $\theta(0) = \pi n_2$ holds. It follows that $n_2 = 2n_0$; in the end

$$n_1 = 2n_\pi - n_2 = 2(n_\pi - n_0) \in 2\mathbb{Z}.$$

A more intrinsic way to show that $w(\det(W_k)) \in 2\mathbb{Z}$ is the following. Due to 2π-periodicity of $\theta'(k)$ we can compute $w(\det(W_k))$ in the interval $[-\pi, +\pi]$ as

$$w(\det(W_k)) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \theta'(k) dk$$

and noting that

$$\int_{-\pi}^{0} \theta'(k) dk = -\int_{\pi}^{0} \theta'(-\kappa) d\kappa = \int_{0}^{\pi} \theta'(-\kappa) d\kappa = \int_{0}^{\pi} \theta'(\kappa) d\kappa,$$

we conclude that

$$w(\det(W_k)) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \theta'(k) dk = 2 \left(\frac{1}{2\pi} \int_{0}^{+\pi} \theta'(k) dk \right) \in 2\mathbb{Z}$$

since

$$\frac{1}{2\pi} \int_{0}^{+\pi} \theta'(k) dk = \frac{1}{2\pi i} \int_{0}^{+\pi} \frac{\partial_k \det(W_k)}{\det(W_k)} dk$$

is a winding number in half circumference due to the condition $\det(W_0) = \det(W_\pi)$. □

Therefore, whenever the physical system admits a symmetry represented, at the level of Hilbert space, by an anti-linear operator such that its square is $-1_{\mathbb{C}^{2m}}$, the Berry phase is even and the previously defined $\mathbb{Z}_2$ invariant turns out to be identically zero. In this case, the invariant does not read topological triviality but instead the presence of a particular structure: the quaternionic one. It follows that the $\mathbb{Z}_2$ invariant is actually influenced also by non-topological structures; only when a quaternionic structure is not present, therefore the invariant is not identically zero, it contains topological information that reports a trivial or non-trivial topological phase. Note, however, that the vanishing of $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ does not necessarily imply the presence of a quaternonic structure. In fact, if $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ is zero, there are two possibilities: either there is a quaternionic structure or the system is in the topologically trivial phase. Since the symmetry operators of the theory are generally known, it is relatively simple to discard the hypothesis of the presence of a quaternionic structure. Conversely, as already noted, if a quaternionic structure is present, the invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ loses its ability to report the presence of a non-trivial topological phase.

4.4 Focus on the BDI symmetry class

We have shown that the invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ also takes into account the quaternionic structure of the system under consideration: if such a structure is present the invariant vanishes. Among the symmetry classes with all three discrete symmetry operators discussed in Section 3.2, the only one without quaternionic structure is the BDI class. In this case, the invariant $\mathrm{I}_{TCS}^{(\mathrm{BDI})}(H)$ records a bit of the topology of the system; in fact the Altland-Zirnbauer classification classifies the BDI chain by a $\mathbb{Z}$-valued invariant while the Berry phase returns the invariant $\mathrm{I}_{TCS}^{(\mathrm{BDI})}(H) \in \mathbb{Z}_2$. The aim of this section is to show, at least within an explicit example, that the invariant $\mathrm{I}_{TCS}^{(\mathrm{BDI})}(H)$ takes into account the parity of the topological invariant given by the Altland-Zirnbauer classification.

4.4.1 Arbitrary distant nearest neighbors Kitaev chain

We consider a slight generalization of Kitaev's chain. The original model was proposed by Kitaev in 2000 [64] as simplified model for a topological superconductor. In our generalization we consider an Hamiltonian H_{Kit} with fibered Hamiltonians given as follows:

$$H_k = \begin{bmatrix} 0 & \sum_{n\in\mathbb{Z}} c_n e^{ink} \\ \sum_{n\in\mathbb{Z}} c_n^* e^{-ink} & 0 \end{bmatrix}, \tag{4.22}$$

and with symmetry operators given by

- time reversal symmetry is given by $\mathfrak{T} = \mathcal{K}$;
- particle hole symmetry is given by $\mathfrak{C} = \sigma_3\mathcal{K}$;
- chiral symmetry is given by $\mathfrak{S} = \sigma_3$.

Such symmetry operators impose the conditions $c_n^* = c_n$. This model takes into account interactions with arbitrarily distant nearest neighbors and we can impose $c_n = 0$ if $|n| > R$ to fit the condition of finite-range hopping model. The Bloch bands are given by

$$\begin{aligned} E_\pm(k) &= \pm\sqrt{\left(\sum_{n\in\mathbb{Z}} c_n e^{ink}\right)\left(\sum_{n\in\mathbb{Z}} c_n^* e^{-ink}\right)} = \\ &= \pm\sqrt{\left(\sum_{n\in\mathbb{Z}} c_n \cos(nk)\right)^2 + \left(\sum_{n\in\mathbb{Z}} c_n \sin(nk)\right)^2} = \\ &= \pm|z(k)| \end{aligned} \tag{4.23}$$

with $z(k) := \sum_{n\in\mathbb{Z}} c_n e^{ink}$. Solving the eigenvalue equation $H_k v(k) = E_-(k)v(k)$ yields the normalized eigenvector,

$$v(k) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ \sqrt{\frac{z^*(k)}{z(k)}} \end{bmatrix}, \tag{4.24}$$

from which we find, with a straightforward computation,

$$\langle v(k), \partial_k v(k)\rangle = \frac{1}{4}\frac{(\partial_k z(k))^* z(k) - z^*(k)(\partial_k z(k))}{z^*(k)z(k)} = -\frac{i}{2}\mathrm{Im}\left(\frac{\partial_k z(k)}{z(k)}\right); \tag{4.25}$$

hence, the $\mathbb{Z}_2$ invariant can be computed as

$$\begin{aligned} \mathrm{I}^{(\mathrm{BDI})}_{TCS}(H) &= \left[\frac{1}{\pi i}\int_{S^1}\langle v(k), \partial_k v(k)\rangle dk\right] = \left[-\frac{1}{2\pi}\int_{S^1}\mathrm{Im}\left(\frac{\partial_k z(k)}{z(k)}\right)dk\right] = \\ &= \left[\mathrm{Re}\left(-\frac{1}{2\pi i}\int_{S^1}\frac{\partial_k z(k)}{z(k)}dk\right)\right]. \end{aligned} \tag{4.26}$$

By the Cauchy integral formula, the integral above computes the winding number around the origin of the curve Γ in the complex plane obtained as the image of $k \mapsto z(k)$. This winding number is an integer, and therefore the real part is redundant. The topological information about topological phases is stored into the complex function $z(k)$ trough its winding number. The invariant $\mathrm{I}^{(\mathrm{BDI})}_{TCS}(H_{\mathrm{Kit}})$ reads the parity of the winding number $w(z(k))$.

Chapter 5

Conclusion

The study of topological phases of matter has unveiled deep and unexpected connections between quantum physics, geometry and topology. This thesis explores such connections, moving from the general framework of Quantum Mechanics to the refined structure of topological invariants and their equivariant versions. Along this path tools from functional analysis, bundle theory and K-theory have played a fundamental role in bridging concepts from pure mathematics with physical phenomena in Condensed Matter Physics. The main goal was to understand whether the Berry phase of a condensed matter system contains topological information and how complete this information is compared to the ten-fold way classification.

In Chapter 1, we provided a functional analytic foundation for the mathematical formulation of quantum theory. We reviewed the basic notions of quantum states and observables, highlighting the role of Hilbert spaces and self-adjoint operators. The spectral theory of unbounded operators, with its two versions of the spectral theorem, was shown to be essential in interpreting physical quantities and in constructing dynamics through the Stone theorem.

In Chapter 2, we shifted the focus toward condensed matter systems, where the interplay between periodicity and quantum mechanics leads to the concept of band theory. Through the Zak-Bloch-Floquet transform and the concept of fibered Hamiltonians, we associated to a Condensed Matter Physics system a vector bundle over the Brillouin torus, known as the Bloch bundle. This bundle encodes the geometry and topology of the occupied states and becomes the natural arena for defining topological invariants. We then introduced and studied vector bundles, principal bundles and connections to establishing the mathematical language in which the topological classification of phases can be cast. These structures reveal how global topological properties of quantum states can lead to quantized physical observables.

Chapter 3 was devoted to the study of topological insulators within the framework of K-theory. We introduced the main discrete symmetries and their algebraic structure, culminating in the ten-fold way classification of symmetry classes. The classification is encoded in a periodic table of topological phases, where each entry is determined by a K-group associated with the space of gapped Hamiltonians compatible with the symmetry constraints. This elegant classification shows the power of abstract algebraic topology in capturing physical properties that are robust under continuous deformations, namely topological invariants. At the same time, it raises subtle questions about how these invariants can be computed directly looking at the Hamiltonian of the system. In this direction, the thesis, based on recent papers, proposes to look at the Berry phase investigating if it is a relevant quantity that contains topological information (and how much) with respect to the ten-fold way classification.

The **original contribution** of this thesis lies in Chapter 4, where we investigated fully symmetric (with all the main discrete symmetries present) chains belonging to the BDI, DIII, CI, and CII symmetry classes and with finitely many internal degrees of freedom. We started from a detailed analysis of the symmetry operators and their action on the fibered Hamiltonians; the actions are reported in Table 4.1. From these actions, in Section 4.2, we defined and constructed symmetric Bloch bases using the tools of parallel transport; specifically, in Theorem 4.2.2, we showed the existence of a symmetric Bloch bases. This allowed us to define the Berry phase and to show that this quantity transforms in a well defined way under a change of symmetric Bloch basis, thanks to Theorem 4.3.3 and the technology of winding numbers for unitary matrices. The starting point to define a topological invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ with values in $\mathbb{Z}_2$ is Theorem 4.3.4, which ensures the possibility to take the $2\mathbb{Z}$ quotient and get a gauge invariant quantity. Furthermore, Theorem 4.3.5 shows the topological invariance of $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$. However, in defining the invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ we have not considered the typical structure of each symmetry class, that is, which discrete symmetry operators squares to $\pm 1_{\mathbb{C}^{2m}}$. This fact has been taken into account in Paragraph 4.3.4 where we showed how in presence of a quaternionic structure, i.e. an anti-linear operator squaring to $-1_{\mathbb{C}^{2m}}$, the invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ has to be vanishing. Therefore, whenever the physical system admits a symmetry represented, at the level of Hilbert space, by an anti-linear operator such that its square is $-1_{\mathbb{C}^{2m}}$, the Berry phase is even and the previously defined $\mathbb{Z}_2$ invariant turn out to be identically zero. In this case, the invariant does not read the topologically trivial phase but instead the presence of a particular structure: the quaternionic one. It follows that the $\mathbb{Z}_2$ invariant is actually influenced also by non-topological structures; only when a quaternionic structure is not present, therefore the invariant is not identically zero, it contains topological information that reports a trivial or non-trivial topological phase. Note, however, that the vanishing of $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ does not necessarily imply the presence of a quaternonic structure. In fact, if $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ is zero there are two possibilities: either there is a quaternion structure or the system is in the topologically trivial phase. This partly answers the initial question about how much topology the Berry phase can expose: in the case of systems with quaternionic structure the invariant loses its effective ability to detect topological phases. Furthermore, the only symmetry class, among the fully symmetric ones that does not have quaternionic structure is the BDI class. However, the ten-fold way classification labels the BDI class with a $\mathbb{Z}$ invariant and not with a $\mathbb{Z}_2$ invariant. In Section 4.4, we analyzed generalized Kitaev chain with arbitrary range couplings in the BDI class and we underlined how $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ could be interpreted as the parity of the $\mathbb{Z}$ invariant provided by the ten-fold way classification.

In summary, this thesis answers the questions formulated in the introduction:

> *Can the Berry phase be used to detect and classify topological phases? If so, how much topology does the Berry phase have access to? To what extent does Berry phase invariant coincide with the invariants predicted by K-theory? Does the Berry phase record the presence of other structures?*

The Berry phase gives rise to the gauge and topological invariant $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$, constructed taking the mod 2 of the Berry phase, which can be used to potentially detect and classify topological phases of matter. However, the Berry phase knows only a little about topology and topological phases since, in the best of all worlds, it is only able to read the parity of the topological invariants reported in periodic table. Furthermore, the Berry phase knows something richer such as the presence of a quaternionic structure of the physical system we are looking.

Future directions may include a general proof of the fact that $\mathrm{I}_{TCS}^{(\mathrm{AZ-class})}(H)$ computes the parity of the topological invariant given by the ten-fold way classification and the investigation of other objects and quantities that can play that the role that was not achieved by the Berry phase.

Appendix A

Gelfand-Naimark-Segal construction

The Gelfand-Naimark-Segal (GNS) construction [8, 65] allows to gain insight onto different representations of a given algebra and it is based on defining a state as a normalized positive linear functional. We start defining C*-algebras in some steps.

Definition A.0.1 ($*$-algebra)**.** *An unital algebra $\mathcal{A}$ with unit element $\mathcal{I}$ is a $*$-algebra if it is equipped with an involution $*$ such that*

1. $(A^*)^* = A \qquad \forall\ A \in \mathcal{A}$;
2. $(AB)^* = B^* A^* \qquad \forall\ A, B \in \mathcal{A}$;
3. $\mathcal{I}^* = \mathcal{I}$.

We can give more properties to $*$-algebras such as the Banach property.

Definition A.0.2 (Banach $*$-algebra)**.** *A Banach $*$-algebra $\mathcal{A}$ is an associative $*$-algebra on the complex or real field that at the same time is also a Banach space, that is a normed space complete in the metric induced by the norm.*

A very important example of Banach $*$-algebras are given by the following objects.

Definition A.0.3 (C*-algebra)**.** *A* C**-algebra is a Banach $*$-algebra which satisfies the* C* *identity* $|A^* A| = |A^*||A| = |A|^2$.

C*-algebras are the key objects for the developments of algebraic techniques in the realm of Quantum Mechanics and beyond; very powerful theorems can be proven for C*-algebras. Let us start with the GNS construction which is a correspondence between cyclic $*$-representations of $\mathcal{A}$ and certain linear functionals on $\mathcal{A}$ (called states). Let us consider a C*-algebra $\mathcal{A}$ and a linear functional $\phi : \mathcal{A} \to \mathbb{C}$; we can define a state.

Definition A.0.4 (State)**.** *A linear functional $\phi : \mathcal{A} \to \mathbb{C}$ with norm given by $|\phi| := |\phi(\mathcal{I})| < +\infty$ and such that for all $A, B \in \mathcal{A}$ it satisfies*

1. $|\phi(AB)|^2 \leq \phi(AA^*)\phi(BB^*)$;

2. $\phi(A^*) = \overline{\phi(A)}$*;*

3. $\phi(A^*A) \geq 0$*;*

4. $|\phi| = 1$*;*

is called a state on $\mathcal{A}$*.*

If we assume that the self-adjoint elements of $\mathcal{A}$ are self-adjoint operators on an Hilber space $\mathcal{H}$ corresponding to physical observables, and that $\mathcal{I} \equiv 1_{\mathcal{H}}$ corresponds to the trivial observable, then for any physical state the linear functional ϕ can be interpreted as an expectation functional over physical observables. The following one is a fundamental definition for the GNS construction.

Definition A.0.5 ($*$-representation)**.** *Let* $\mathcal{A}$ *be a* C**-algebra, a* $*$*-representation of* $\mathcal{A}$ *on a Hilbert space* $\mathcal{H}$ *is a mapping* π *from* $\mathcal{A}$ *into the algebra of bounded operators on* $\mathcal{H}$ *such that it is a ring homomorphism and unit preserving.*

Each positive linear functional ϕ over a C*-algebra $\mathcal{A}$ defines a Hilbert space $\mathcal{H}_\phi$ and a $*$-representation π_ϕ of $\mathcal{A}$ by linear operators. Since $\mathcal{A}$ is a linear space over the field $\mathbb{C}$ then ϕ defines a hermitian semidefinite product on $\mathcal{A}$ defined as

$$\langle A|B\rangle := \phi(B^*A)$$

and such that

$$\langle A|A\rangle \geq 0, \qquad |\langle A|B\rangle|^2 \leq \langle A|A\rangle \langle B|B\rangle\,, \tag{A.1}$$

for all $A, B \in \mathcal{A}$. In order to eliminate the semidefinite property and to ensure the non-degeneracy of the hermitian product we need the following result.

Proposition A.0.1 (Gelfand ideal)**.** *The set* $\mathcal{J} := \{X \in \mathcal{A} \mid \phi(X^*X) = 0\} \subset \mathcal{A}$ *is a left ideal and it is called the Gelfand ideal of the state.*

Proof. Let Y be an element of $\mathcal{A}$ and X and element of $\mathcal{J}$; we need to show that $YX \in \mathcal{J}$. In fact let us call $B = (X^*Y^*Y)^*$ and $A = X$ then

$$\begin{aligned} |\phi(X^*Y^*YX)|^2 = |\phi(B^*A)|^2 = |\langle A|B\rangle|^2 \leq \langle A|A\rangle \langle B|B\rangle = \\ = \langle X|X\rangle \langle (X^*Y^*Y)^*|(X^*Y^*Y)^*\rangle = 0 \end{aligned}$$

since $X \in \mathcal{J}$. Hence we have that $|\phi(X^*Y^*YX)|^2 \leq 0$ therefore must hold that $|\phi(X^*Y^*YX)|^2 = 0$ and so $\phi(X^*Y^*YX) = 0$. □

An element ψ in the quotient $\mathcal{A}/\mathcal{J}$ corresponds to the equivalence class $[A]$ of the elements of $\mathcal{A}$ modulo $\mathcal{J}$ with $\langle\psi|\psi\rangle = |\psi|^2 > 0$: in this way we obtain a linear space equipped with hermitian positive definite scalar product. By completing this space we get the Hilbert space $\mathcal{H}_\phi$. Note that the product (A.1) does not depend on $[A]$. The action of the $*$-representation $\pi_\phi(A) : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi)$ on $\mathcal{H}_\phi$ is then

$$\pi_\phi(A)(\psi) = [AB] \qquad \text{if} \;\; \psi = [B]. \tag{A.2}$$

Let us now move on to the following Definition.

Definition A.0.6 (Cyclic vector and cyclic representation)**.** *A vector $\xi \in \mathcal{H}$ is cyclic if given a $*$-representation π, the subspace $\tilde{\mathcal{H}} := \{\tilde{\xi} \in \pi(A)\xi\}$ is dense in $\mathcal{H}$. A $*$-representation π is cyclic if a cyclic vector $\xi \in \mathcal{H}$ exists.*

Furthermore we can define the state

$$\phi(A) := \langle \xi | \pi_\phi(A) | \xi \rangle \,, \tag{A.3}$$

which is sometimes referred to as the vacuum state; in a similar way any vector $\psi \in \mathcal{H}_\phi$ defines a state by

$$\phi_\psi(A) := \langle \psi | \pi_\phi(A) | \psi \rangle \,. \tag{A.4}$$

We can now state GNS theorem whose proof with any details can be found in [66].

Theorem A.0.1 (GNS theorem)**.** *Given a* C*-*algebra $\mathcal{A}$ endowed with unit and a linear positive functional ϕ on $\mathcal{A}$, then the triple $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ consisting of an Hilbert space, a $*$-representation and a ciclic vector, exists and is unique up to unitarity equivalence. Moreover,*

$$\phi(A) = \langle \xi | \pi_\phi(A) | \xi \rangle \,.$$

An interesting corollary of the entire costruction is that if ϕ is a pure state, i.e. for every pair of states ϕ_1, ϕ_2 on $\mathcal{A}$ and every $\lambda \in (0,1)$ the implication

$$\phi = \lambda\phi_1 + (1-\lambda)\phi_2 \quad \Rightarrow \quad \phi_1 = \phi_2 = \phi$$

holds[1], then π_ϕ is an irreducible $*$-representation, i.e. the only closed invariant subspaces are the entire Hilbert space and $\{0\}$. The GNS construction is the key result for the Algebraic Quantum Mechanics, that is the formulation of quantum theory in terms of C*-algebras of observables, and for the definition of non-unitarily equivalent formulations of Quantum Mechanics.

[1] That is, ϕ cannot be written as a non-trivial convex combination of other states: it is an extreme point of the convex compact set of all states.

Appendix B

Proof of Wigner's Theorem

Here we give a complete proof of the Wigner's Theorem 1.3.1.

Proof. Let us first prove that any unitary or anti-unitary U induces a Wigner symmetry transformation T_U through

$$T_U([\psi]) := [U(\psi)].$$

Indeed, for every $\psi, \phi \in \mathcal{H}$ we have $[\psi'] := T_U([\psi]) = [U(\psi)]$ and $[\phi'] := T_U([\phi]) = [U(\phi)]$; therefore

$$|\langle\phi'|\psi'\rangle| = |\langle U\phi|U\psi\rangle| = |\widetilde{\langle\psi|\phi\rangle}| = |\langle\psi|\phi\rangle|$$

regardless of whether U is unitary or anti-unitary since $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$ and $|\widetilde{\langle\psi|\phi\rangle}| \in \mathbb{R}$.

On the other hand, from the condition that a symmetry has to preserve the probability transition we can construct explicitly the operator that acts on the physical state, which turns out to be unitary or anti-unitary. We follow Weinberg proof. We are interested in symmetry transformations T that preserve transition probabilities:

$$|(\psi_1', \psi_2'\rangle|^2 = |\langle\psi_1, \psi_2\rangle|^2.$$

Let $\{\psi_k\}$ be an orthonormal set

$$\langle\psi_k, \psi_l\rangle = \delta_{kl};$$

the transformed vector ψ_k' also form an orthonormal set, since

$$|\langle\psi_k', \psi_l'\rangle|^2 = |\langle\psi_k, \psi_l\rangle|^2 = \delta_{kl}.$$

We define

$$\phi_k := \frac{1}{\sqrt{2}}(\psi_k' + \psi_1'),$$

with $\psi_k' := U(\psi_k)$ and $\psi_1' := U(\psi_1)$; then

$$U\left(\frac{1}{\sqrt{2}}(\psi_k + \psi_1)\right) = \frac{1}{\sqrt{2}}(U(\psi_k) + U(\psi_1)) = \phi_k.$$

Now, any arbitrary vector ψ can be expanded as:

$$\psi = \sum_k C_k \psi_k,$$

and its transformed version as

$$\psi' = \sum_k C'_k U(\psi_k),$$

with $\{C_k\}$ and $\{C'_k\}$ complex numbers. From the preservation of inner products with ψ_k and ϕ_k, we obtain, respectively

$$\begin{aligned} |C_k|^2 &= |C'_k|^2, \qquad \forall\, k \\ |C_k + C_1|^2 &= |C'_k + C'_1|^2, \qquad \forall\, k \neq 1; \end{aligned}$$

taking the ratio of the above equations and looking again at the first one, we deduce that

$$\begin{aligned} \mathrm{Re}\left(\frac{C_k}{C_1}\right) &= \mathrm{Re}\left(\frac{C'_k}{C'_1}\right), \\ \mathrm{Im}\left(\frac{C_k}{C_1}\right) &= \pm \mathrm{Im}\left(\frac{C'_k}{C'_1}\right). \end{aligned}$$

Hence we have

$$\frac{C_k}{C_1} = \frac{C'_k}{C'_1}, \quad \text{or} \quad \frac{C_k}{C_1} = \left(\frac{C'_k}{C'_1}\right)^*.$$

The choice must hold for all k as we can show by supposing that the ratios are complex and that the first holds for a specific k and the second holds for a $l \neq k$: we will arrive to a contradiction since then the ratios $\frac{C_k}{C_1}$ and $\frac{C_l}{C_1}$ will be real. This dichotomy leads to two distinct possibilities for the operator U

$$\begin{aligned} U\left(\sum_k C_k \psi_k\right) &= \sum_k C_k U \psi_k \quad \text{(linear)}, \\ U\left(\sum_k C_k \psi_k\right) &= \sum_k C_k^* U \psi_k \quad \text{(anti-linear)}. \end{aligned}$$

It remains to be proved that for a given symmetry transformation, we must make the same choice between the two above for arbitrary values of the coefficients C_k. Suppose that the first one applies for a vector $\sum_k A_k \psi_k$ while the second one applies for a vector $\sum_k B_k \psi_k$. Then the invariance of transition probabilities requires that

$$\left|\sum_k B_k^* A_k\right|^2 = \left|\sum_k B_k A_k^*\right|^2,$$

or equivalently

$$\sum_{k,l} \mathrm{Im}(A_k^* A_l)\mathrm{Im}(B_k^* B_l) = 0.$$

We cannot rule out the possibility that the above equation may be satisfied for a pair of vectors $\sum_k A_k\psi_k$ and $\sum_k B_k\psi_k$ with different global phase. However, for any pair of such vectors, with neither A_k nor B_k all of the same phase, we can always find a third vector $\sum_k C_k\psi_k$ for which

$$\sum_{k,l} \mathrm{Im}(C_k^* C_l)\mathrm{Im}(A_k^* A_l) \neq 0,$$

and also

$$\sum_{k,l} \mathrm{Im}(C_k^* C_l)\mathrm{Im}(B_k^* B_l) \neq 0.$$

Hence the same choice of U linear or anti-linear must be made for $\sum_k A_k\psi_k$ and $\sum_k C_k\psi_k$, and the same choice must be made for $\sum_k B_k\psi_k$ and $\sum_k C_k\psi_k$, so the same choice must also be made for the two vectors $\sum_k A_k\psi_k$ and $\sum_k B_k\psi_k$ with which we started. We have thus shown that for a given symmetry transformation T either all vectors satisfy linearity or else they all satisfy anti-linearity. It is now easy to show that as we have defined it, the quantum mechanical operator U is either linear and unitary or anti-linear and anti-unitary. First, suppose that U is linear for all vectors. Any two vectors ψ and ϕ may be expanded as

$$\psi = \sum_k A_k\psi_k, \quad \phi = \sum_k B_k\phi_k$$

and so

$$\begin{aligned} U(\alpha\psi + \beta\phi) &= U\left(\sum_k (\alpha A_k\psi_k + \beta B_k\phi_k)\right) = \sum_k (\alpha A_k U(\psi_k) + \beta B_k U(\phi_k)) \\ &= \alpha \sum_k A_k U(\psi_k) + \beta \sum_k B_k U(\phi_k) = \alpha U(\psi) + \beta U(\phi) \end{aligned}$$

so U is in fact linear. Moreover, using the orthonormality, the scalar product of the transformed states is

$$\langle U\psi, U\phi\rangle = \sum_{k,l} A_k^* B_l \langle U(\psi_k), U(\phi_l)\rangle = \sum_k A_k^* B_k = \langle \psi, \phi\rangle,$$

so U is unitary. In a similar way we can show that if U is anti-linear we get an anti-unitary U. $\square$

Appendix C

Existence and uniqueness of the horizontal lift and parallel transport properties

Theorem C.0.1 (Existence and uniqueness of the horizontal lift)**.** *Let $\gamma : [0,1] \to X$ be a curve in X and let $u_0 \in \pi^{-1}(\gamma(0))$. Then there exists a unique horizontal lift $\tilde{\gamma}(t)$ in P such that $\tilde{\gamma}(0) = u_0$.*

Proof. Let U_i be a trivializing open containing the image of γ, and take a section s_i over U_i. If there exists a horizontal lift $\tilde{\gamma}$, it may be expressed as

$$\tilde{\gamma}(t) = s_i(\gamma(t))\, g_i(t),$$

where $g_i(t) := g_i(\gamma(t)) \in \mathrm{G}$. Without loss of generality, we may choose the section such that $s_i(\gamma(0)) = \tilde{\gamma}(0)$, i.e., $g_i(0) = e$.

Let $W_{\gamma(0)}$ be a tangent vector to $\gamma(t)$ at $\gamma(0)$. Then $\tilde{W}_{u_0} = \tilde{\gamma}_*(W_{\gamma(0)})$ is tangent to $\tilde{\gamma}$ at $u_0 := \tilde{\gamma}(0)$. Since the tangent vector $\tilde{W}_{u_0}$ is horizontal, it satisfies $\omega(\tilde{W}_{u_0}) = 0$. Therefore we can show that

$$\tilde{W}_{u_0} = g_i(t)^{-1}((s_i)_*(W_{\gamma(0)}))g_i(t) + \left[g_i(t)^{-1} dg_i(W_{\gamma(0)})\right]^{\#};$$

applying ω to both sides, we find:

$$0 = \omega(\tilde{W}_{u_0}) = g_i(t)^{-1}\omega((s_i)_*(W_{\gamma(0)}))g_i(t) + g_i(t)^{-1}\frac{dg_i(t)}{dt}.$$

Multiplying on the left by $g_i(t)$, we obtain the ODE

$$\frac{dg_i(t)}{dt} = -\omega((s_i)_*(W_{\gamma(0)}))g_i(t).$$

The fundamental theorem of ODEs guarantees the existence and uniqueness of the solution of the above differential equation. Since $\omega((s_i)_*X) = (s_i)^*\omega(X) = A_i(X)$, the above ODE takes the local form

$$\frac{dg_i(t)}{dt} = -A_i(W_{\gamma(0)})\, g_i(t),$$

whose formal solution with $g_i(0) = e$ is given by

$$g_i(\gamma(t)) = \mathcal{P}\left[\exp\left(-\int_0^t (A_i)_\mu \frac{dx^\mu}{dt} dt\right)\right] = \mathcal{P}\left[\exp\left(-\int_{\gamma(0)}^{\gamma(t)} (A_i)_\mu(\gamma(t)) dx^\mu\right)\right],$$

where $\mathcal{P}[\cdot]$ denotes the path-ordering operator along $\gamma(t)$. The horizontal lift is therefore expressed as

$$\tilde{\gamma}(t) = s_i(\gamma(t)) g_i(\gamma(t)).$$

□

Proposition C.0.1 (Properties of parallel transport). *Let $\tilde{\gamma}$ be a horizontal lift of $\gamma : [0,1] \to X$ and let $\Gamma(\tilde{\gamma})$ the parallel transport map. Then*

1. *the parallel transport map commutes with the right action*

$$R_g \circ \Gamma(\tilde{\gamma}) = \Gamma(\tilde{\gamma}) \circ R_g;$$

2. *consider the map*

$$\Gamma(\tilde{\gamma}^{-1}) : \pi^{-1}(\gamma(1)) \to \pi^{-1}(\gamma(0))$$

defined by the curve $\tilde{\gamma}^{-1}(t) := \tilde{\gamma}(1-t)$. Then we have

$$\Gamma(\tilde{\gamma}^{-1}) = \Gamma(\tilde{\gamma})^{-1};$$

3. *consider two curves $\alpha : [0,1] \to X$ and $\beta : [0,1] \to X$ such that $\alpha(1) = \beta(0)$. Define the concatenated curve $\alpha * \beta(t) : [0,1] \to X$ by*

$$\alpha * \beta := \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Let

$$\Gamma(\tilde{\alpha}) : \pi^{-1}(\alpha(0)) \to \pi^{-1}(\alpha(1)), \quad \Gamma(\tilde{\beta}) : \pi^{-1}(\beta(0)) \to \pi^{-1}(\beta(1))$$

be the parallel transports along α and β, respectively. Then the parallel transport along $\alpha * \beta$ satisfies:

$$\Gamma(\widetilde{\alpha * \beta}) = \Gamma(\tilde{\beta}) \circ \Gamma(\tilde{\alpha}).$$

Proof. Let $P(X, \mathrm{G})$ be a principal G-bundle with a connection, and let ω be the associated connection 1-form on P. Let $\tilde{\gamma} : [0,1] \to P$ be the horizontal lift of a curve $\gamma : [0,1] \to X$ with initial point $\tilde{\gamma}(0) = u_0 \in P$ and end point $\tilde{\gamma}(1) = u_1 \in P$, so that $\pi \circ \tilde{\gamma} = \gamma$.

1. Let $u \in P$ and $g \in \mathrm{G}$. Since $\tilde{\gamma}$ is horizontal, we have $\omega(\dot{\tilde{\gamma}}_{\tilde{\gamma}(t)}(t)) = 0$ for all t. Consider the curve $\tilde{\gamma}^{(g)}(t) := \tilde{\gamma}(t) g$ and compute

$$\frac{d}{dt} \tilde{\gamma}^{(g)}(t) = (R_g)_*(\dot{\tilde{\gamma}}(t)).$$

Due to the properties of ω we have,

$$\omega\left(\dot{\tilde{\gamma}}^{(g)}_{\tilde{\gamma}^{(g)}(t)}(t)\right) = ((R_g)^*(\omega))(\dot{\tilde{\gamma}}_{\tilde{\gamma}(t)}(t)) = \mathrm{Ad}_{g^{-1}}(\omega(\dot{\tilde{\gamma}}_{\tilde{\gamma}(t)}(t))) = 0.$$

So $\tilde{\gamma}^{(g)}$(t) is horizontal for all t. Moreover, since we have that $\tilde{\gamma}^{(g)}(0) = \tilde{\gamma}(0)g = u_0g$, it is the horizontal lift of γ starting at u_0g. Hence:

$$\Gamma(\tilde{\gamma})(R_g(u_0)) = \Gamma(\tilde{\gamma})(u_0g) = \tilde{\gamma}(1)g = \Gamma(\tilde{\gamma})(u_0)g = R_g(\Gamma(\tilde{\gamma})(u_0)).$$

Therefore:

$$R_g \circ \Gamma(\tilde{\gamma}) = \Gamma(\tilde{\gamma}) \circ R_g.$$

2. Let $\tilde{\gamma}^{-1}(t) := \tilde{\gamma}(1-t)$. We claim that $\tilde{\gamma}^{-1}$ is horizontal. Indeed:

$$\frac{d}{dt}\tilde{\gamma}^{-1}(t) = -\dot{\tilde{\gamma}}_{\tilde{\gamma}(1-t)}(1-t),$$

so

$$\omega\left(\dot{\tilde{\gamma}}^{-1}_{\tilde{\gamma}^{-1}(1-t)}(t)\right) = -\omega(\dot{\tilde{\gamma}}_{\tilde{\gamma}(1-t)}(1-t)) = 0.$$

Thus $\tilde{\gamma}^{-1}$ is a horizontal lift of the reversed curve $\gamma^{-1}(t) = \gamma(1-t)$ starting at $\tilde{\gamma}(1)$. Then $\Gamma(\tilde{\gamma}^{-1})$ represent the inverse parallel transport of $\Gamma(\tilde{\gamma})$ since

$$\Gamma(\tilde{\gamma}^{-1})(\tilde{\gamma}^{-1}(0)) = \Gamma(\tilde{\gamma}^{-1})(u_1) = \tilde{\gamma}^{-1}(1) = \tilde{\gamma}(0) = u_0$$

and therefore

$$(\Gamma(\tilde{\gamma}^{-1}) \circ \Gamma(\tilde{\gamma}))(u_0) = \Gamma(\tilde{\gamma}^{-1})(u_1)) = u_0,$$
$$(\Gamma(\tilde{\gamma}) \circ \Gamma(\tilde{\gamma}^{-1}))(u_1) = \Gamma(\tilde{\gamma})(u_0)) = u_1.$$

Therefore:

$$\Gamma(\tilde{\gamma}^{-1}) = \Gamma(\tilde{\gamma})^{-1}.$$

3. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be horizontal lifts of α and β with matching endpoints: $\tilde{\alpha}(1) = \tilde{\beta}(0)$. Define a curve $\widetilde{\alpha * \beta}$ by:

$$\widetilde{\alpha * \beta}(t) = \begin{cases} \tilde{\alpha}(2t), & 0 \le t \le \frac{1}{2}, \\ \tilde{\beta}(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

We verify that $\widetilde{\alpha * \beta}$ is smooth and horizontal, being composed of horizontal curves. We have that

$$\Gamma(\widetilde{\alpha * \beta})(\tilde{\alpha}(0)) = \Gamma(\widetilde{\alpha * \beta})(\widetilde{\alpha * \beta}(0)) = \widetilde{\alpha * \beta}(1) = \tilde{\beta}(1)$$

and

$$\Gamma(\tilde{\beta}) \circ \Gamma(\tilde{\alpha})(\tilde{\alpha}(0)) = \Gamma(\tilde{\beta})(\tilde{\alpha}(1)) = \Gamma(\tilde{\beta})(\tilde{\beta}(0)) = \tilde{\beta}(1).$$

Hence

$$\Gamma(\widetilde{\alpha * \beta}) = \Gamma(\tilde{\beta}) \circ \Gamma(\tilde{\alpha}).$$

□

Appendix D

Curvature and Chern classes

In this Appendix we introduce the concepts of curvature of a bundle and of Chern classes.

Curvature of a G-bundle

This object measures how much the connection deviates from being trivial in the sense that vectors transported parallel along a closed curve are unchanged. Hence, the curvature contains all the local information about how the bundle "twists". Moreover, in the physical contexts the curvature of a bundle is of capital importance: in physics, it is related to forces. For example, the Riemann curvature tensor in General Relativity [67, 68] is the curvature of the tangent bundle, or the force field in a gauge theory with gauge group G, the so-called field strength, is related to the curvature of a principal G-bundle. Therefore, the curvature describes all interactions in the physical world [68, 45].

Definition D.0.1 (Curvature 2-form)**.** *Let $P(X, \mathrm{G})$ be a* G*-bundle with connection 1-form ω. Then the curvature 2-form is defined*

$$\Omega := d_P\omega + \omega \wedge \omega$$

and acts on $W_u, Z_u \in T_uP$ as

$$\Omega(W_u, Z_u) = d_P\omega(W_u, Z_u) + \omega \wedge \omega(W_u, Z_u)$$

where d_P is the exterior derivative on P and $\omega \wedge \omega(W_u, Z_u) = [\omega(W_u), \omega(Z_u)]_{\mathfrak{g}}$.

The local form of the curvature is defined by

$$F_i := (s_i)^*(\Omega) \tag{D.1}$$

where s_i is a local section defined on a trivializing open chart $U_i \subset M$ (recall that $A_i = (s_i)^*(\omega)$). The curvature is expressed in terms of the gauge potential as

$$F_i = dA_i + A_i \wedge A_i, \tag{D.2}$$

that is known as Yang-Mills field strength. A connection is said to be flat if its curvature vanishes; in physics language, a gauge field configuration with vanishing

field strength is known as pure gauge configuration.
Under a change of local trivializations with trivializing opens U_i and U_j the field strength must satisfy an appropriate condition.

Proposition D.0.1 (Field strength compatibility relation)**.** *Let $\{U_i\}$ be an open covering of X and $g_{U_iU_j}(x)$ the transition function at point $x \in X$. Then the Yang-Mills field strength satisfies the compatibility relation*

$$F_j = g^{-1}_{U_iU_j} F_i g_{U_iU_j}.$$

Proof. Starting with

$$F_j = dA_j + A_j \wedge A_j$$

and using the local transformation property (2.52) for the gauge potential we have

$$\begin{aligned} dA_j &= d(g^{-1}_{U_iU_j} A_i g_{U_iU_j} + g^{-1}_{U_iU_j} dg_{U_iU_j}) = \\ &= dg^{-1}_{U_iU_j} \wedge A_i g_{U_iU_j} + g^{-1}_{U_iU_j} dA_i g_{U_iU_j} + g^{-1}_{U_iU_j} A_i \wedge dg_{U_iU_j} + dg^{-1}_{U_iU_j} \wedge g_{U_iU_j}, \end{aligned} \tag{D.3}$$

and

$$A_j \wedge A_j = (g^{-1}_{U_iU_j} A_i g_{U_iU_j} + g^{-1}_{U_iU_j} dg_{U_iU_j}) \wedge (g^{-1}_{U_iU_j} A_i g_{U_iU_j} + g^{-1}_{U_iU_j} dg_{U_iU_j}). \tag{D.4}$$

Using the relation $dg^{-1}_{U_iU_j} = -g^{-1}_{U_iU_j} dg_{U_iU_j} g^{-1}_{U_iU_j}$ we complete the proof. □

Chern classes

The Chern classes are characteristic classes[1] associated with complex vector bundles; they are topological invariants. Chern classes provide a simple test for bundles isomorphism: if the Chern classes of a pair of complex vector bundles do not agree then the bundles are different. The converse, however, is not true. Moreover, Chern classes are fundamental for some theorems of capital importance in many areas of mathematics: the (Hirzebruch)-Riemann-Roch theorem, the Chern-Gauss-Bonnet theorem and more generally the Atiyah–Singer index theorem. We now present Chern classes informally. Given a complex vector bundle $\mathcal{E} = (E, \pi, X)$ over a topological space X, the Chern classes of $\mathcal{E}$ are a sequence of elements of the cohomology of X. The k-th Chern class of $\mathcal{E}$, denoted $c_k(E)$, is an element of $H^{2k}(X; \mathbb{Z})$, the cohomology of X with integer coefficients. For example if X is a smooth manifold then the Chern classes have representative in the de Rham cohomology groups. The sum of all non-vanishing Chern classes

$$c(E) := \sum_{k=0}^{\dim_{\mathbb{C}}(X)} c_k(E), \tag{D.5}$$

is known as total Chern class. A Chern polynomial is a convenient way to handle Chern classes and related notions systematically. By definition, for a complex vector

[1]In general a characteristic class is a way of associating to each principal G-bundle of X a cohomology class of X.

bundle $\mathcal{E}$, the Chern polynomial of $\mathcal{E}$ is given by

$$c_t(E) := \sum_{k=0}^{\dim_{\mathbb{C}}(X)} c_k(E)t^k. \tag{D.6}$$

This is not a new invariant since the formal variable t simply keeps track of the degree of $c_k(E)$. In particular, $c_t(E)$ is completely determined by the total Chern class and conversely. Before moving on we need to define an operation that makes cohomology groups, rings.

Definition D.0.2 (Cup product)**.** *In cohomology, the cup product is a construction that defines a product on the $H^\bullet(X)$ of a topological space X. If α^p is a p-cochain and β^q is a q-cochain, then the cup product is defined on a singular $(p+q)$-simplex σ by*

$$(\alpha^p \smile \beta^q)(\sigma) := \alpha^p(\sigma \circ \iota_{0,1,\ldots,p}) \cdot \beta^q(\sigma \circ \iota_{p,p+1,\ldots,p+q}),$$

where ι_S, for $S \subset \{0,1,\ldots,p+q\}$, denotes the canonical embedding of the simplex spanned by S into the standard $(p+q)$-simplex with vertices $\{0,\ldots,p+q\}$.

The cup product satisfies the following identity

$$\delta(\alpha^p \smile \beta^q) = \delta\alpha^p \smile \beta^q + (-1)^p(\alpha^p \smile \delta\beta^q), \tag{D.7}$$

where $\delta : H^\bullet(X) \to H^{\bullet+1}(X)$ is the codifferential map. From the above identity, it follows that the cup product of two cocycles is again a cocycle, and the product of a coboundary with a cocycle (in either order) is a coboundary. Hence, the cup product induces a well-defined bilinear operation on cohomology, $\smile : H^p(X) \times H^q(X) \to H^{p+q}(X)$. We can now give the axiomatic definition of Chern classes.

Definition D.0.3 (Chern classes)**.** *Given a complex vector bundle $\mathcal{E} = (E, \pi, X)$, Chern classes $c_k(E)$ are elements of $H^{2k}(X;\mathbb{Z})$ that satisfy the following four axioms:*

1. *Normalization of the zeroth Chern class axiom. For any complex vector bundle $\mathcal{E} = (E, \pi, X)$ we have $c_0(E) = 1$;*

2. *Naturality axiom. Let $f : Y \to X$ be a continuous map and let $\mathcal{F}^*\mathcal{E} = (f^*E, \Pi, Y)$ be the pullback of the complex vector bundle $\mathcal{E} = (E, \pi, X)$ then $c_k(f^*E) = f^*c_k(E)$;*

3. *Whitney sum axiom. Let $\mathcal{E}' = (E', \pi', X)$ be another complex vector bundle, then the Chern classes of the Whitney sum bundle $\mathcal{E} \oplus \mathcal{E}' = (E \oplus E', \Pi, X)$ satisfy $c(E \oplus E') = c(E) \smile c(E')$, i.e. $c_k(E \oplus E') = \sum_{i=0}^{k} c_i(E) \smile c_{k-i}(E')$;*

4. *Normalization axiom. The total Chern class of the tautological line bundle over $\mathbb{PC}^k$ is given by $c(\mathcal{O}(-1)) = 1 - H$, where H is the Poincaré dual of the hyperplane $\mathbb{PC}^{k-1} \subseteq \mathbb{PC}^k$.*

Since Chern classes are cohomology elements, they encode topological properties of the complex vector bundle, but they encode also curvature related properties of the bundle since Chern classes can be presented via the curvature 2-form of a connection on the complex vector bundle using the Chern-Weil construction. Chern-Weil theory computes topological invariants of $\mathbb{K}$-vector bundles and principal G-bundles on a smooth manifold X in terms of connections and curvature representing classes in the de Rham cohomology rings of X. The theory forms a bridge between the areas of algebraic topology and differential geometry. The goal of Chern-Weil theory is to construct cohomology classes which:

1. depend only on the topological structure of the bundle;
2. are obtained from any connection on the bundle;
3. are represented by closed differential forms constructed from the curvature of the chosen connection.

The construction begins considering a Lie group $(\mathrm{G}, \cdot)$, its Lie algebra $(\mathfrak{g}.[\cdot,\cdot]_{\mathfrak{g}})$ and the homogeneous polynomials $f : \mathfrak{g} \to \mathbb{K}$ of degree k.

Definition D.0.4 (Subring $\mathbb{K}[\mathfrak{g}]^{\mathrm{G}}$)**.** *Let* $(\mathrm{G}, \cdot)$ *be a Lie group, let* $(\mathfrak{g}.[\cdot,\cdot]_{\mathfrak{g}})$ *be its Lie algebra with* $\dim(\mathfrak{g}) = k$ *and let* $\mathbb{K}$ *be a field. Let* $\mathbb{K}[\mathfrak{g}]$ *be the polynomial ring formed by taking polynomials in the variables that represent the Lie algebra generators* $\mathfrak{X}_1, ...\mathfrak{X}_k$*, then we define the subring of fixed points of* $\mathbb{K}[\mathfrak{g}]$ *under the adjoint action of Lie group* $(\mathrm{G}, \cdot)$ *as*

$$\mathbb{K}[\mathfrak{g}]^{\mathrm{G}} := \{f \in \mathbb{K}[\mathfrak{g}] \mid f(\mathrm{Ad}_{g^{-1}}(x_1\mathfrak{X}_1 + ... + x_k\mathfrak{X}_k)) = f(x_1\mathfrak{X}_1 + ... + x_k\mathfrak{X}_k)\},$$

where $g \in \mathrm{G}, x_1, ..., x_k \in \mathbb{K}$ and $\mathfrak{X}_1, ..., \mathfrak{X}_k \in \mathfrak{g}$.

We can apply $f \in \mathbb{K}[\mathfrak{g}]^{\mathrm{G}}$ homogeneous of degree k to the curvature 2-form Ω to get the scalar-valued $2k$-form on P, $f(\Omega)$, given by

$$f(\Omega)(v_1, \ldots, v_{2k}) := \frac{1}{(2k)!} \sum_{\sigma \in \mathfrak{S}_{2k}} \mathrm{sign}(\sigma) \tilde{f}(\Omega(v_{\sigma(1)}, v_{\sigma(2)}), \ldots, \Omega(v_{\sigma(2k-1)}, v_{\sigma(2k)})), \tag{D.8}$$

where v_i with $i \in \{1, ..., 2k\}$ are tangent vectors to P. Definition (D.8) takes into account the polarization $\tilde{f} : \mathfrak{g}^k \to \mathbb{K}$ of the invariant homogeneous polynomial $f \in \mathbb{K}[\mathfrak{g}]^{\mathrm{G}}$ which is a multilinear and symmetric functional on $\mathfrak{g}^k$ that inherits the invariance property; moreover since $f(\Omega)$ must be a $2k$-form we need to antisymmetrize. The following Theorem is crucial in the Chern-Weil constuction.

Theorem D.0.1 (Chern-Weil)**.** *Let* $f \in \mathbb{K}[\mathfrak{g}]^{\mathrm{G}}$ *be an invariant homogeneous polynomial of degree* k*. Then, the form* $f(\Omega)$ *on* P *descends to a unique, closed form* $\overline{f(\Omega)}$ *on* X*; i.e., there is a closed form on* X *whose pullback is* $f(\Omega)$*. Moreover, the de Rham cohomology class of* $\overline{f(\Omega)}$ *on* X *is independent of the choice of a connection.*

Proof. We first prove that a differential form α on P descends to a unique form $\bar{\alpha}$ on X, i.e. $\pi^*(\bar{\alpha}) = \alpha$, if and only if α is horizontal and invariant, that is:

- it vanishes whenever at least one argument is vertical, i.e. tangent to the fiber;
- it satisfies $(R_g)^*(\alpha) = \alpha$ for all $g \in \mathrm{G}$.

On the one hand, if $\alpha = \pi^*(\bar{\alpha})$ then α vanishes on vertical vectors; indeed let $W_u \in T_uP$ be vertical, this means that $\pi_*(W_u) = 0$. Then we have

$$\alpha(W_u, \dots) = (\pi^*(\bar{\alpha}))(W_u, \dots) = \bar{\alpha}(\pi_*(W_u), \dots) = 0$$

since at least one of the arguments is zero. Moreover, the condition (2.45), $\pi \circ R_g = \pi$, implies invariance since

$$(R_g)^*(\alpha) = (R_g)^*(\pi^*(\bar{\alpha})) = (\pi \circ R_g)^*(\bar{\alpha}) = \pi^*(\bar{\alpha}) = \alpha.$$

On the other hand, given a horizontal and invariant form α, define

$$\bar{\alpha}|_{U_i} := (s_i)^*(\alpha),$$

for a local section $s_i : U_i \to P$. To conclude we need to show that this definition is independent of the choice of the section s_i, that $\bar{\alpha}|_{U_i}$ extend to all of X and that $\pi^*(\bar{\alpha}) = \alpha$. Let s_i, s_j be two local sections over an open set $U_i \cap U_j \subseteq X$; then there exists a smooth function $g_{U_jU_i} : U_i \cap U_j \to \mathrm{G}$ such that:

$$s_j(x) = s_i(x) g_{U_iU_j}(x).$$

Since α is G-invariant, we have:

$$(s_j)^*(\alpha) = (R_{g_{U_iU_j}(x)} \circ s_i)^*(\alpha) = (s_i)^*((R_{g_{U_iU_j}(x)})^*(\alpha)) = (s_i)^*(\alpha),$$

which shows that the form defined in this way is well defined on $U_i \cap U_j$. At this point, since in the intersections we have $(\bar{\alpha}|_{U_i})|_{U_i \cap U_j} = (\bar{\alpha}|_{U_j})|_{U_i \cap U_j}$, the sheaf properties make possible that local forms glue to a global form $\bar{\alpha}$ on X satisfying $\pi^*(\bar{\alpha}) = \alpha$.
We now need to prove that $f(\Omega)$ is horizontal and G-invariant to conclude that $\overline{f(\Omega)}$ exist and its unique.
For the horizontal property we compute, for a point $u \in P$,

$$\Omega(\mathfrak{X}^{\#}_u, Y_u) = d\omega(\mathfrak{X}^{\#}_u, Y_u) + [\omega(\mathfrak{X}^{\#}_u), \omega(Y_u)]_{\mathfrak{g}};$$

to show that the curvature 2-form vanishes on the vertical vector fields since all vertical vector fields can be constructed using the map $\#$. Using $\omega(\mathfrak{X}^{\#}_u) = \mathfrak{X}$, and the formula for the exterior derivative we get

$$d\omega(\mathfrak{X}^{\#}_u, Y_u) = \mathfrak{X}^{\#}_u(\omega(Y_u)) - Y(\omega(\mathfrak{X}^{\#}_u)) - \omega([\mathfrak{X}^{\#}_u, Y_u]_{\mathfrak{g}}).$$

Inserting in the curvature 2-form definition we get

$$\Omega(\mathfrak{X}^{\#}_u, Y_u) = \mathfrak{X}^{\#}_u(\omega(Y_u)) - Y_u(\omega(\mathfrak{X}^{\#}_u)) - \omega([\mathfrak{X}^{\#}_u, Y_u]_{\mathfrak{g}}) + [\mathfrak{X}, \omega(Y_u)]_{\mathfrak{g}}.$$

Now, the term $Y_u(\omega(\mathfrak{X}^{\#}_u))$ is vanishing since

$$\omega(\mathfrak{X}^{\#}_u) = \mathfrak{X} \qquad \forall\ u \in P$$

hence we have to do the derivative of a constant. To compute $\mathfrak{X}^{\#}_u(\omega(Y_u))$ we define the function

$$h(t) := \omega(Y_{u \exp(t\mathfrak{X})}) \in \mathfrak{g};$$

Therefore, by definition 2.3.8, we have

$$\mathfrak{X}_u^{\#}(\omega(Y_u)) = \frac{d}{dt}h(t)\Big|_{t=0} = \frac{d}{dt}\omega\left(Y_{u\exp(tX)}\right)\Big|_{t=0}.$$

Thanks to the defining equivariant property of the connection 1-form we have

$$((R_g)^*(\omega))(W_u) = \mathrm{Ad}_{g^{-1}}\left(\omega((R_g)_*(W_u))\right)$$

and applying this with $g = \exp(t\mathfrak{X})$ we get

$$\omega\left(Y_{u\exp(t\mathfrak{X})}\right) = \left((R_{\exp(t\mathfrak{X})})^*\omega\right)(Y_u) = \mathrm{Ad}_{\exp(-tX)}\left(\omega\left((R_{\exp(tX)})_*(Y_u)\right)\right).$$

Now, taking the derivative at $t = 0$ we have

$$\mathfrak{X}_u^{\#}[\omega(Y_u)] = \frac{d}{dt}\mathrm{Ad}_{\exp(-t\mathfrak{X})}\left(\omega\left((R_{\exp(t\mathfrak{X})})_*(Y_u)\right)\right)\Big|_{t=0};$$

using the identities

$$\frac{d}{dt}\mathrm{Ad}_{\exp(-t\mathfrak{X})}(Z_u)\Big|_{t=0} = -[\mathfrak{X}, Z_u]_{\mathfrak{g}}, \quad \frac{d}{dt}\Big|_{t=0}\omega((R_{\exp(t\mathfrak{X})})_*(Y_u)) = \omega([\mathfrak{X}_u^{\#}, Y_u]_{\mathfrak{g}}),$$

we get

$$\mathfrak{X}_u^{\#}(\omega(Y_u)) = -[\mathfrak{X}, \omega(Y_u)]_{\mathfrak{g}} + \omega([\mathfrak{X}_u^{\#}, Y_u]_{\mathfrak{g}}).$$

In the end we have

$$\Omega(\mathfrak{X}_u^{\#}, Y_u) = 0,$$

for every $Y_u \in T_uP$; so the curvature 2-form is horizontal. Now looking at the definition (D.8) we conclude that also $f(\Omega)$ is horizontal.
For the G-invariant property we show that the curvature form Ω satisfies the following equivariance property under the right action of the structure group G:

$$(R_g)^*(\Omega) = \mathrm{Ad}_{g^{-1}}(\Omega) \qquad \forall\ g \in \mathrm{G}$$

This property follows directly from the definition of the curvature and the G-equivariance of the connection form. Indeed, we have

$$\Omega = d\omega + [\omega \wedge \omega],$$

and applying the pullback by the right action we get

$$(R_g)^*(\Omega) = (R_g)^*(d\omega) + (R_g)^*([\omega\wedge\omega]) = d((R_g)^*(\omega)) + [(R_g)^*(\omega) \wedge (R_g)^*(\omega)]$$

since the exterior derivative commutes with pullback and bracket of forms is bilinear. We now use the G-equivariance of the connection form,

$$(R_g)^*(\omega) = \mathrm{Ad}_{g^{-1}}(\omega),$$

so

$$\begin{aligned} &d((R_g)^*(\omega)) = d(\mathrm{Ad}_{g^{-1}}(\omega)) = \mathrm{Ad}_{g^{-1}}(d\omega), \\ &[(R_g)^*(\omega) \wedge (R_g)^*(\omega)] = [\mathrm{Ad}_{g^{-1}}(\omega) \wedge \mathrm{Ad}_{g^{-1}}(\omega)] = \mathrm{Ad}_{g^{-1}}[(\omega\wedge\omega)]. \end{aligned}$$

In the end

$$(R_g)^*(\Omega) = \mathrm{Ad}_{g^{-1}}(d\omega) + \mathrm{Ad}_{g^{-1}}([\omega \wedge \omega]) = \mathrm{Ad}_{g^{-1}}(d\omega + [\omega \wedge \omega]) = \mathrm{Ad}_{g^{-1}}(\Omega).$$

Moreover, due to the invariance of f, we have

$$(R_g)^*(f(\Omega)) = f(\Omega).$$

This proves that $f(\Omega)$ descends to a unique form on X we call $\overline{f(\Omega)}$.
It is the turn of the closure property. Since $f(\Omega) = \pi^*\left(\overline{f(\Omega)}\right)$, to conclude that $d\overline{f(\Omega)} = 0$ it is enough to show that $df(\Omega) = 0$ since exterior derivative commutes with the pullback. The closure property of $f(\Omega)$ follows from the invariance of the polynomial f and from the Bianchi identity $d\Omega + [\omega, \Omega]_{\mathfrak{g}} = 0$. Since $\tilde{f}$ is multilinear and symmetric, the exterior derivative obeys the Leibniz rule

$$\begin{aligned} &df(\Omega)(v_0, ..., v_{2k}) = \\ &= \sum_{\sigma,i} \frac{\mathrm{sign}(\sigma)}{(2k)!} \tilde{f}(\Omega(v_{\sigma(1)}, v_{\sigma(2)}), \ldots, d\Omega(v_{\sigma(2i-1)}, v_{\sigma(2i)}, v_{\sigma(0)}), \ldots, \Omega(v_{\sigma(2k)}, v_{\sigma(2k)})) + \\ &\quad + \text{terms with } v_{\sigma(0)} \mapsto v_{\sigma(j)}; \end{aligned}$$

for all $j \in [1, 2k]$; where $d\Omega$ appears in the i-th slot, $\sigma \in \mathfrak{S}_{2k}$ and $i \in [1, k]$. The Bianchi identity states that

$$d\Omega + [\omega, \Omega] = 0;$$

so we have, schematically, terms of the form

$$\sum_{i=1}^{k} \tilde{f}(\Omega(v_{\sigma(1)}, v_{\sigma(2)}), \ldots, [\Omega, \omega](v_{\sigma(2i-1)}, v_{\sigma(2i)}, v_{\sigma(0)}), \ldots, \Omega(v_{\sigma(2k)}, v_{\sigma(2k)})).$$

The key point now is that if $\tilde{f}$ is an invariant multilinear symmetric functional on $\mathfrak{g}^k$ we have

$$\sum_{i=1}^{k} \tilde{f}(\mathfrak{X}_1, \ldots, [Y, \mathfrak{X}_i], \ldots, \mathfrak{X}_k) = 0,$$

where $\mathfrak{X}_1, \ldots, \mathfrak{X}_k, Y \in \mathfrak{g}$. In fact, since $\tilde{f}$ is invariant under the adjoint action of G we have

$$\tilde{f}(\mathrm{Ad}_{g^{-1}} \mathfrak{X}_1, \ldots, \mathrm{Ad}_{g^{-1}} \mathfrak{X}_k) = \tilde{f}(\mathfrak{X}_1, \ldots, \mathfrak{X}_k);$$

differentiating at $t = 0$ the above formula along the curve $g^{-1}(t) := \exp(tY) \in \mathrm{G}$, we get

$$\sum_{i=1}^{k} \tilde{f}(\mathfrak{X}_1, \ldots, [Y, \mathfrak{X}_i], \ldots, \mathfrak{X}_k) = 0$$

as claimed. By multilinearity the same holds also for k generic elements of $\mathfrak{g}$. Taking $Y = \omega$ and all the others elements as Ω we get

$$\sum_{i=1}^{k} \tilde{f}(\Omega(v_{\sigma(1)}, v_{\sigma(2)}), \ldots, [\Omega, \omega](v_{\sigma(2i-1)}, v_{\sigma(2i)}, v_{\sigma(0)}), \ldots, \Omega(v_{\sigma(2k)}, v_{\sigma(2k)})) = 0;$$

therefore $df(\Omega) = 0$.
The last point is the independence from the connection. Let ω_0 and ω_1 be two connections on P with curvature forms Ω_0 and Ω_1. Consider the product bundle $(P \times [0,1], \pi \times 1_{[0,1]}, X \times [0,1])$ and let P_t the total space of this bundle. We define

$$\omega_t := (1-t)p^*(\omega_0) + tp^*(\omega_1)$$

as a connection on P_t where $p : P_t \to P$ be the projection. Let Ω_t be the associated curvature. The invariant polynomial f gives us $f(\Omega_t)$ as $2k$-form on P_t which descends to a unique form $\overline{f(\Omega_t)}$ on $X \times [0,1]$. Let

$$\iota_s : X \to X \times [0,1], \quad x \mapsto (x,s)$$

be the inclusions; then ι_0 is homotopic to ι_1. Thus, $(\iota_0)^* \left(\overline{f(\Omega_t)}\right)$ and $(\iota_1)^* \left(\overline{f(\Omega_t)}\right)$ belong to the same de Rham cohomology class by the homotopy invariance of de Rham cohomology. Finally, by naturality and by uniqueness of descending,

$$(\iota_0)^* \left(\overline{f(\Omega_t)}\right) = \overline{f(\Omega_0)} \quad \text{and} \quad (\iota_1)^* \left(\overline{f(\Omega_t)}\right) = \overline{f(\Omega_1)}.$$

Therefore they are cohomologous

$$\left[\overline{f(\Omega_1)}\right] = \left[\overline{f(\Omega_0)}\right] \in H^{2k}_{\mathrm{dR}}(X;\mathbb{K}).$$

This completes the poof. $\square$

The above theorem is fundamental in view of the writing of Chern classes in terms of the curvature. Thanks to the above result we can give the following Definition.

Definition D.0.5 (Chern-Weil map)**.** *Let $P(X, \mathrm{G})$ be a principal G-bundle. The Chern-Weil map is*

$$\mathrm{CW}_E : \mathbb{K}[\mathfrak{g}]^{\mathrm{G}} \to H^{2k}(X;\mathbb{K}), \quad f \mapsto \left[\overline{f(\Omega)}\right],$$

which sends invariant homogeneous polynomials of degree k to cohomology classes of degree k.

The following proposition shows as the Chern-Weil map is a ring homomorphism.

Proposition D.0.2. CW_E *is a ring homomorphism, hence*

$$\mathrm{CW}_E(f+h) = \mathrm{CW}_E(f) + \mathrm{CW}_E(h),$$
$$\mathrm{CW}_E(fh) = \mathrm{CW}_E(f) \smile \mathrm{CW}_E(h).$$

Proof. Let us start with the additivity preservation. Let f, h two invariant homogeneous polynomials of degree k, then $f+h$ is an invariant homogeneous polynomials of degree k and we have

$$\overline{(f+h)(\Omega)} = \overline{f(\Omega)} + \overline{h(\Omega)}.$$

since f and h are applied as multilinear symmetric invariant polynomials on $\mathfrak{g}^k$. Since the de Rham cohomology class of a sum of closed forms is the sum of their classes, we have

$$\mathrm{CW}_E(f+h) = \left[\overline{f(\Omega)} + \overline{h(\Omega)}\right] = \left[\overline{f(\Omega)}\right] + \left[\overline{h(\Omega)}\right] = \mathrm{CW}_E(f) + \mathrm{CW}_E(h).$$

Now the multiplication preservation. Let f, h two invariant homogeneous polynomials of degree k and l respectively, then fh is an invariant homogeneous polynomials of degree $k+l$ and we have

$$\overline{(fh)(\Omega)} = \overline{f(\Omega)} \wedge \overline{h(\Omega)},$$

passing to cohomology, since the wedge product of closed forms descends to the cup product in de Rham cohomology, we have

$$\mathrm{CW}_E(fh) = \left[\overline{f(\Omega) \wedge h(\Omega)}\right] = \left[\overline{f(\Omega)}\right] \smile \left[\overline{h(\Omega}\right] = \mathrm{CW}_E(f) \smile \mathrm{CW}_E(h).$$

$\square$

Since we are interested in complex vector bundles with typical fiber $\mathbb{C}^m$ we are going to consider invariant polynomial f on $\mathfrak{gl}(m, \mathbb{C})$ to build up Chern classes. Using a formal parameter t, the total class is defined in terms of the curvature 2-form, which is a $m \times m$ matrix-valued 2-form in the case of a complex vector bundle, using the invariant polynomial given by the determinant

$$c_t(E) := \det\left(I - \frac{t}{2i\pi}\Omega\right) = 1 + c_1(E)t + c_2(E)t^2 + \cdots + c_m(E)t^m; \tag{D.9}$$

where each $c_k(E)$ turns out to be a de Rham cohomology class of degree $2k$, constructed as a gauge-invariant polynomial in the curvature Ω. Our goal is to factorize the determinant, to do so we need the following Theorem due to Grothendieck.

Theorem D.0.2 (Formal Splitting Principle)**.** *Let $\mathcal{E} = (E, \pi, X)$ be a complex vector bundle of rank m. Even though $\mathcal{E}$ may not globally split as a direct sum of line bundles $\mathcal{L}_i = (L_i, \pi, X)$, there exists a ring extension of $H^\bullet(X; \mathbb{C})$ in which $\mathcal{E}$ behaves formally as a direct sum*

$$\mathcal{E} \simeq \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m,$$

where each $\mathcal{L}_i$ is a formal line bundle with first Chern class $x_i := c_1(L_i)$ called i-th formal Chern root. In this ring extension we have

$$c_t(E) := \det\left(I - \frac{t}{2\pi i}\Omega\right).$$

The formal Chern roots are not actual cohomology classes but formal elements in a suitable extension of the cohomology ring. The next theorem gives an important connection between Chern classes and elementary symmetric polynomial we define now.

Definition D.0.6 (Elementary symmetric polynomials)**.** *The elementary symmetric polynomials in m variables $x_1, \dots, x_m$ are defined as*

$$\begin{aligned}
\sigma_0(x) &:= 1, \\
\sigma_1(x) &:= x_1 + x_2 + \cdots + x_m, \\
\sigma_2(x) &:= \sum_{i<j<m} x_i x_j, \\
&\vdots \\
\sigma_k(x) &= \sum_{1 \le i < j < \cdots < l \le m} x_i x_j \cdots x_l, \\
&\vdots \\
\sigma_m(x) &:= x_1 x_2 \cdots x_m.
\end{aligned}$$

The next Theorem gives the connection we are looking for.

Theorem D.0.3. *Let $x_1, \dots, x_m$ be the formal Chern roots of a rank m complex vector bundle (E, π, X). Then the Chern classes $c_k(E)$ are elementary symmetric polynomials in the x_i. Any such symmetric polynomial of degree $k > m$ is identically zero, hence the Chern class $c_k(E)$ vanishes for all $k > m$.*

Proof. By the formal splitting principle, we may formally regard (E, π, X) of rank m as a direct sum of m line bundles with formal first Chern classes $x_1, \dots, x_m$. The k-th Chern class $c_k(E)$ is then the k-th elementary symmetric polynomial in $x_1, \dots, x_m$. Indeed, we can expand the determinant as a polynomials in Chern roots $x_1, \dots, x_m$ which are the formal eigenvalues of the matrix $\tilde{\Omega} := -\frac{1}{2\pi i}\Omega$

$$c_t(E) := \det\left(I - \frac{t}{2\pi i}\Omega\right) = \prod_{i=1}^{m}(1 + t x_i).$$

Expanding we get

$$c_t(E) = \prod_{i=1}^{m}(1 + t x_i) = 1 + \sigma_1(x)t + \sigma_2(x)t^2 + \cdots + \sigma_m(x)t^m,$$

where the $\sigma_k(x)$ with $k \in [1, m]$ is the k-th elementary symmetric polynomial in the Chern roots $x_1, ..., x_m$. We can give the explixit case for $m = 3$. The Chern polynomial is

$$\begin{aligned}
c_t(E) &= (1 + t x_1)(1 + t x_2)(1 + t x_3) = \\
&= 1 + (x_1 + x_2 + x_3)t + (x_1 x_2 + x_1 x_3 + x_2 x_3)t^2 + (x_1 x_2 x_3)t^3.
\end{aligned}$$

Therefore

$$\begin{aligned}
c_1(E) &= x_1 + x_2 + x_3 = \sigma_1(x), \\
c_2(E) &= x_1 x_2 + x_1 x_3 + x_2 x_3 = \sigma_2(x), \\
c_3(E) &= x_1 x_2 x_3 = \sigma_3(x).
\end{aligned}$$

In the same manner, we conclude the general case. We immediately see that since Ω is a $m \times m$ matrix its determinant contains at most powers of order m in t, so from the general definition of the Chern polynomial we get that $c_k(E) = 0$ if $k > m$. Another way is to see that the space of elementary symmetric polynomials in m variables has dimension m, and no non-zero elementary symmetric polynomial of degree $k > m$ exists. $\square$

Therefore, the Chern class $c_k(E)$ is given by the elementary symmetric polynomial $\sigma_k(x_1, \dots, x_m)$ in the Chern roots. We give the full explicit computation for a complex vector bundle of rank 2. Let

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

the curvature matrix and define

$$\widetilde{\Omega} := -\frac{1}{2\pi i}\Omega.$$

We compute

$$\det\Big(I + t\widetilde{\Omega}\Big) = \det\begin{pmatrix} 1 + t\widetilde{\omega}_{11} & t\widetilde{\omega}_{12} \\ t\widetilde{\omega}_{21} & 1 + t\widetilde{\omega}_{22} \end{pmatrix} = (1 + t\widetilde{\omega}_{11})(1 + t\widetilde{\omega}_{22}) - (\widetilde{\omega}_{12}\widetilde{\omega}_{21})t^2;$$

expanding we get

$$\det\Big(I + t\widetilde{\Omega}\Big) = 1 + \widetilde{\omega}_{11}t + \widetilde{\omega}_{22}t + \widetilde{\omega}_{11}\widetilde{\omega}_{22}t^2 - \widetilde{\omega}_{12}\widetilde{\omega}_{21}t^2.$$

Thus

$$c_1(E) = \widetilde{\omega}_{11} + \widetilde{\omega}_{22} = \sigma_1(x) = \operatorname{tr}(\widetilde{\Omega}),$$

and

$$c_2(E) = \widetilde{\omega}_{11}\widetilde{\omega}_{22} - \widetilde{\omega}_{12}\widetilde{\omega}_{21} = \sigma_2(x) = \det\Big(\widetilde{\Omega}\Big).$$

The above example leads to the following Proposition.

Proposition D.0.3 (Newton's identities)**.** *Let $x_1, ..., x_m$ be variables and let $p_k(x)$ be, for $k \geq 1$, the k-th power sum*

$$p_k(x) := \sum_{i=1}^{m} x_i^k = x_1^k + \cdots + x_m^k.$$

Then

$$k\sigma_k(x) = \sum_{i=1}^{k} (-1)^{i-1}\sigma_{k-i}(x)p_i(x) \qquad \forall\ m \geq k \geq 1;$$

where $\sigma_k(x)$ is the k-th elementary symmetric polynomial in variables $x_1, ..., x_m$. Moreover, if $x_1, ..., x_m$ are the formal Chern roots, then the symmetric polynomials $\{\sigma_0(x), ..., \sigma_m(x)\}$ can be written as polynomial expressions with rational coefficients in $\operatorname{tr}(\widetilde{\Omega}), ..., \operatorname{tr}(\widetilde{\Omega}^m)$

Proof. We aim to prove the formula by induction; the $k = 1$ case is simple:

$$\sigma_1(x) = p_1(x)$$

by definition. For the general case we assume the identity holds for k and we are going to prove it for $k + 1$. We define the total symmetric polynomial

$$E(t) := \sum_{k=0}^{m} e_k(x)t^k;$$

and we have

$$\frac{d}{dt}E(t) = E(t)\sum_{i=1}^{m}\frac{x_i}{1+tx_i}$$

which follows from differentiating $\prod_{i=1}^{m}(1 + tx_i)$ with respect to t and applying the product rule. Formally expanding we get

$$\begin{aligned}\sum_{i=1}^{m}\frac{x_i}{1+tx_i} &= \sum_{i=1}^{m}\sum_{j=1}^{\infty}(-1)^{j-1}x_i^j t^{j-1} = \\ &= \sum_{j=1}^{\infty}(-1)^{j-1}\left(\sum_{i=1}^{m}x_i^j\right)t^{j-1} = \sum_{j=1}^{\infty}(-1)^{j-1}p_j(x)t^{j-1},\end{aligned}$$

where we changed the order of summation. In the end

$$E'(t) = E(t)\sum_{j=1}^{\infty}(-1)^{j-1}p_j(x)t^{j-1}.$$

Now by comparing the coefficients of t^k on both sides and changing label $i \mapsto k+1-j$ in the right hand side, we get

$$(k+1)e_{k+1}(x) = \sum_{i=1}^{k+1}(-1)^{i-1}e_{k+1-i}(x)p_i(x).$$

Newton's identities can be inverted to express $\sigma_k(x)$ in therms of power sums as

$$e_k(x) = (-1)^k \sum_{\substack{j_1+2j_2+\cdots+kj_k=k \\ j_1\geq 0,\ldots,j_k\geq 0}} \prod_{i=1}^{k}\frac{(-p_i(x))^{j_i}}{j_i!\, i^{j_i}};$$

this result con be shown by induction.
We now note that if we have a matrix $\tilde{\Omega}$ with eigenvalues $\{x_i\}_{i\in[1,m]}$, then the matrix $\tilde{\Omega}^k$ has eigenvalues $\{x_i^k\}_{i\in[1,m]}$. Therefore we have the relation

$$p_k(x) = \operatorname{tr}(\tilde{\Omega}^k).$$

In conclusion, we get that the elementary symmetric polynomial, and so the Chern classes, can be expressed as polynomial expressions with rational coefficients in traces of powers of $\tilde{\Omega}$. $\square$

Appendix E

Classification of vector bundles

In this section we study the classification up to isomorphism of complex vector bundles. Let X be a topological manifold. Let $\mathcal{L} = (L, \pi, X)$ be a line bundle, i.e. a complex vector bundle of rank 1. Define

$$S \subseteq H^0(X, L)$$

as a vector subspace of the space of sections, with $n = \dim(S) \leq \dim(H^0(X, L))$. We note that if X is compact, then $H^0(X, L)$ is finite dimensional and we can simply take $S = H^0(X, L)$. For each point $x \in X$ we define the evaluation map

$$\mathrm{ev}_x : S \to L_x, \quad s \mapsto s(x)$$

where L_x is the fiber of L at x. Since we must have

$$\mathrm{Ker}(\mathrm{ev}_x) \subseteq S \quad \text{and} \quad \dim(\mathrm{Range}(\mathrm{ev}_x)) \subseteq \dim(L_x) = 1,$$

we have two possibilities:

1. $\dim(\mathrm{Range}(\mathrm{ev}_x)) = 0$;
2. $\dim(\mathrm{Range}(\mathrm{ev}_x)) = 1$.

The first one motivates the following Definition.

Definition E.0.1 (Base point). *Let S be a finite dimensional vector subspace of the space of sections $H^0(X, L)$. A point $x \in X$ is said to be a base point for S if for all $s \in S$ we have $s(x) = 0$, i.e. if* $\dim(\mathrm{Range}(\mathrm{ev}_x)) = 0$.

We note that if $\dim(\mathrm{Range}(\mathrm{ev}_x)) = 1$, there is at least one section in S not vanishing at x and so x is not a base point. If S has no base points, then for all $x \in X$ we have $\mathrm{Ker}(\mathrm{ev}_x) \subset S$ and we get a morphism

$$\Phi_S : X \to \{\text{hyperplanes of } S\} = \mathrm{Gr}_{n-1,n}(S) \simeq \mathrm{Gr}_{1,n}(S^\vee) = \mathbb{P}(S^\vee).$$

Over $\mathbb{P}(S^\vee)$ there is the tautological line bundle $\mathcal{O}_{\mathbb{P}(S^\vee)}$; the following result states that if S has no base points, $\mathcal{L}$ can be realized as the pullback of $\mathcal{O}_{\mathbb{P}(S^\vee)}$ via Φ_S.

Theorem E.0.1. *Let $\mathcal{L} = (L, \pi, X)$ be a complex line bundle over X without base points and let $\mathcal{O}_{\mathbb{P}(S^\vee)}$ be the tautological line bundle over $\mathbb{P}(S^\vee) = \mathrm{Gr}_{1,n}(S^\vee)$. Then*

$$\mathcal{L} \simeq (\Phi_S)^* \left(\left(\mathcal{O}_{\mathbb{P}(S^\vee)} \right)^\vee \right)$$

Proof. We now prove that given $x \in X$ we have

$$L_x \simeq (\tilde{\Phi}_S)^*((E_{\mathrm{tau}})_x)^\vee$$

where $(\tilde{\Phi}_S)^*$ is the pullback map on the fibers. Be definition, we have

$$(\tilde{\Phi}_S)^* \left(((E_{\mathrm{tau}})_x)^\vee \right) = ((E_{\mathrm{tau}})^\vee)_{\tilde{\Phi}_S(x)} = ((E_{\mathrm{tau}})^\vee)_{[\varphi]}$$

where φ is a functional on S such that $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\mathrm{ev}_x)$. Since if V is a complex vector space there is an isomorphsim between the sections of the tautological bundle over $\mathrm{Gr}_{1,n}(V) = \mathbb{P}(V)$ and the linear functional on V we get that

$$((E_{\mathrm{tau}})^\vee)_{[\varphi]} = (\mathbb{C}\varphi)^\vee.$$

Now let us consider the map

$$\varepsilon : S \to (\mathbb{C}\varphi)^\vee, \quad s \mapsto \mathrm{ev}_x(s).$$

We have

- $\mathrm{Ker}(\varepsilon) = \{s \in S \mid \mathrm{ev}_x(s) = 0\} = \mathrm{Ker}(\mathrm{ev}_x)$;
- $\mathrm{Range}(\varepsilon) = S/\mathrm{Ker}(\varepsilon) = S/\mathrm{Ker}(\mathrm{ev}_x) = \mathrm{Range}(\mathrm{ev}_x) = L_x$;

follows that $\dim(\mathrm{Range}(\varepsilon)) = 1$ and so must be isomorphic to $(\mathbb{C}\varphi)^\vee$. Therefore $(\mathbb{C}\varphi)^\vee \simeq L_x$ and the proof is complete. $\square$

Hence, if $\mathcal{L}$ has enough sections, so as not to have base points, then it is obtained as a pullback, via a map from the basis X to a projective space of sufficient dimension, of the dual of the tautological bundle over that projective space. In the smooth complex case, this is always possible thanks to partitions of unity. In the case of a complex vector bundle of rank m the situation is similar. Requiring that it have sufficient frames of m sections then the evaluation map is surjective and we can repeat the same construction to obtain the following result.

Theorem E.0.2. *Let $\mathcal{E} = (E, \pi, X)$ be a complex vector bundle of rank m over X and let S be a n-dimensional vector subspace of $H^0(X, E)$. Let us assume X without base points and let $\mathcal{O}_{\mathrm{Gr}_{m,n}(S^\vee)}$ be the tautological vector bundle over $\mathrm{Gr}_{m,n}(S^\vee)$. Then*

$$\mathcal{E} \simeq (\Phi_S)^* \left(\left(\mathcal{O}_{\mathrm{Gr}_{m,n}(S^\vee)} \right)^\vee \right)$$

where

$$\Phi_S : X \to \{\text{subspaces of dimension } n - m \text{ of } S\} = \mathrm{Gr}_{n-m,n}(S) \simeq \mathrm{Gr}_{m,n}(S^\vee).$$

Proof. The proof is given by similar considerations to those of proof of Theorem E.0.1. □

In the end, every complex vector bundle of rank m over X is isomorphic to the pullback of the dual of the tautological bundle over $\mathrm{Gr}_{m,n}(S^\vee)$ via a continuous map

$$\Phi_S : X \to \mathrm{Gr}_{m,n}(S^\vee).$$

This result be further extended.

Theorem E.0.3 (Homotopy classification)**.** *Let* $\mathrm{Vect}_m(X)$ *denote the set of isomorphism classes of complex vector bundles of rank m over X. There is a bijection*

$$[X, G_{m,\infty}(S_\infty^\vee)] \leftrightarrow \mathrm{Vect}_m(X), \quad [\Phi_{S_\infty}] \mapsto \left[(\Phi_{S_\infty})^* \left(\left(\mathcal{O}_{\mathrm{Gr}_{m,\infty}(S_\infty^\vee)}\right)^\vee\right)\right],$$

where $[X, G_{m,\infty}(S_\infty^\vee)]$ *denotes the set of homotopy classes of smooth maps from X to* $G_{m,\infty}(S_\infty^\vee)$*.*

Proof. Let us consider a sequence of complex vector spaces $S_0^\vee \subset \ ... \subset \ S_{n-1}^\vee \subset S_n^\vee \subset S_{n+1}^\vee \subset \ ... \ \subset S_\infty^\vee$ with dimensions $\dim(\mathfrak{S}^\vee) = k$ and let consider the the natural inclusion

$$S_n^\vee \to S_{n+1}^\vee, \quad v \to (v, 0)$$

where $v \in S_n^\vee$. This induces a map on the corresponding Grassmannians

$$\mathrm{Gr}_{m,n}(S_n^\vee) \xrightarrow{\iota} \mathrm{Gr}_{m,n+1}(S_{n+1}^\vee).$$

We can lift this inclusion to the correspondent tautological bundle, indeed

$$\iota^*(\mathcal{O}_{\mathrm{Gr}_{m,n+1}(S_{n+1}^\vee)})_{[W]} = (\mathcal{O}_{\mathrm{Gr}_{m,n+1}(S_{n+1}^\vee)})_{\iota([W])} \simeq W \oplus 0 = (\mathcal{O}_{\mathrm{Gr}_{m,n}(S_n^\vee)})_{[W]};$$

therefore

$$\iota^*\left(\left(\mathcal{O}_{\mathrm{Gr}_{m,n+1}(S_{n+1}^\vee)}\right)^\vee\right) \simeq \left(\mathcal{O}_{\mathrm{Gr}_{m,n}(S_n^\vee)}\right)^\vee.$$

Consider the following homotopy commutative diagram:

$$\begin{tikzcd} & \mathrm{Gr}_{m,n}(S_n^\vee) \arrow[d, "\iota"] \\ X \arrow[ur, "\Phi_{S_n}"] \arrow[r, "\Phi_{S_{n+1}}"'] & \mathrm{Gr}_{m,n+1}(S_{n+1}^\vee) \end{tikzcd}$$

we have

$$\begin{aligned}(\Phi_{S_n})^*\left(\left(\mathcal{O}_{\mathrm{Gr}_{m,n}(S_n^\vee)}\right)^\vee\right) &\simeq (\Phi_{S_n})^*\left(\iota^*\left(\left(\mathcal{O}_{\mathrm{Gr}_{m,n+1}(S_{n+1}^\vee)}\right)^\vee\right)\right) \\ &\simeq_h (\Phi_{S_{n+1}})^*\left(\left(\mathcal{O}_{\mathrm{Gr}_{m,n+1}(S_{n+1}^\vee)}\right)^\vee\right),\end{aligned}$$

since $(\Phi_{S_n})^* \circ \iota^* = (\iota \circ \Phi_{S_n})^*$ by definition. We now take the direct limit in n to get the infinite Grassmannian $\mathrm{Gr}_{m,\infty}(S_\infty^\vee)$. Therefore, we have a map

$$[X, G_{m,\infty}(S_\infty^\vee)] \leftrightarrow \mathrm{Vect}_m(X), \quad [\Phi_{S_\infty}] \mapsto \left[(\Phi_{S_\infty})^* \left(\left(\mathcal{O}_{\mathrm{Gr}_{m,\infty}(S_\infty^\vee)}\right)^\vee\right)\right].$$

To conclude that it is an isomorphism we note that, due to Theorem 2.3.2, two homotopic maps from X to the infinite Grassmannian leads to isomorphic complex vector bundles and hence the same class in $\mathrm{Vect}_m(X)$. Conversely, we have to show that if the pullbacks of the dual of the tautological bundle over the infinite Grassmannian by two maps are isomorphic, then the two maps are homotopic. This can be done working directly on the local trivializations. $\square$

Theorem E.0.3 teaches us that the invariants under isomorphism of complex vector bundles are the same as the invariants of $[X, G_{m,\infty}(S_\infty^\vee)]$. In general, given a pair of spaces X and Y, we want to construct invariants for $[X, Y]$ by using cohomological tools. Let $H^\bullet(Y;\mathbb{Z})$ be the cohomology ring of Y with integer coefficients (although any coefficient ring would work). A map $f : X \to Y$ induces

$$f^* : H^\bullet(Y;\mathbb{Z}) \to H^\bullet(X;\mathbb{Z}), \tag{E.1}$$

which allows us to define invariants of f by evaluating cohomology classes of Y on X via f. In particular, the induced map f^* only depends on the homotopy class of f; this gives a natural map Φ

$$\Phi : [X, Y] \to \mathrm{Hom}_{\mathbb{Z}}\left(H^{\cdot\bullet}(Y;\mathbb{Z}), H^\bullet(X;\mathbb{Z})\right). \tag{E.2}$$

Now, let us take $Y = \mathrm{Gr}_{m,\infty}(S_\infty^\vee)$, the infinite Grassmannian. Given a class $[c] \in H^\bullet(\mathrm{Gr}_{m,\infty}(S_\infty^\vee);\mathbb{Z})$ and a map $\Phi_{S_\infty} : X \to \mathrm{Gr}_{m,\infty}(S_\infty^\vee)$ such that the complex vector bundle $\mathcal{E}$ over X is isomorphic to the pullback of the dual of the tautological bundle over the infinite Grassmannian, the pullback

$$(\Phi_{S_\infty})^*([c]) \in H^\bullet(X;\mathbb{Z}) \tag{E.3}$$

defines a characteristic class of the complex vector bundle over X. In this way, we associate to each rank m complex vector bundle over X some cohomology classes in $H^\bullet(X;\mathbb{Z})$. Thus, to compute all such invariants, it is enough to understand the ring $H^\bullet(\mathrm{Gr}_{m,\infty}(S_\infty^\vee);\mathbb{Z})$.
The above construction motivates the following Definition.

Definition E.0.2 (Classifying map, classifying space and universal Chern classes)**.** *Let $\mathcal{E} = (E, \pi, X)$ be a complex vector bundle of rank m over X. A map $\Phi_{S_\infty} : X \to \mathrm{Gr}_{m,\infty}(S_\infty^\vee)$ such that the complex vector bundle $\mathcal{E}$ over X is isomorphic to the pullback of the dual of the tautological bundle over the infinite Grassmannian is called the classifying map while the infinite Grassmannian is called the classyfing space. Chern classes $c_i \in H^{2i}(\mathrm{Gr}_{m,\infty}(S_\infty^\vee);\mathbb{Z})$ are called universal Chern classes.*

We now give a corollary of Theorem E.0.3 that we are going to use in the discussion about K-theory.

Corollary E.0.1. *Every complex vector bundle $\mathcal{E}$ of rank m over X can be embedded as a subbundle of a trivial bundle $\mathcal{E}_{\mathrm{tri}}^{(m)}$ over X.*

Proof. Let Φ_{S_∞} a classifying map for $\mathcal{E}$ and define $\mathcal{F} := (\Phi_{S_\infty})^*(\mathcal{Q}_{\mathrm{Gr}_{m,\infty}(S_\infty^\vee)})$. Then

$$\begin{aligned} \mathcal{E} \oplus \mathcal{F} &\simeq (\Phi_{S_\infty})^*(\mathcal{O}_{\mathrm{Gr}_{m,\infty}(S_\infty^\vee)}) \oplus (\Phi_{S_\infty})^*(\mathcal{Q}_{\mathrm{Gr}_{m,\infty}(S_\infty^\vee)}) \simeq \\ &\simeq (\Phi_{S_\infty})^*(\mathcal{O}_{\mathrm{Gr}_{m,\infty}(S_\infty^\vee)} \oplus \mathcal{Q}_{\mathrm{Gr}_{m,\infty}(S_\infty^\vee)}) \simeq \mathcal{E}_{\mathrm{tri}}^{(m)}. \end{aligned} \tag{E.4}$$

since $\mathcal{O}_{\mathrm{Gr}_{m,\infty}(S_\infty^\vee)} \oplus \mathcal{Q}_{\mathrm{Gr}_{m,\infty}(S_\infty^\vee)}$ is isomorphic to the trivial bundle over the infinite Grassmannian. The same result holds also for finite Grassmannians. $\square$

Definition E.0.2 can be extended to the case of a general principal G-bundle over a CW-complex topological space X.

Definition E.0.3 (Universal principal bundle)**.** *Let* G *be a topological group. A universal principal* G*-bundle is a principal* G*-bundle* $E\mathrm{G}(B\mathrm{G},\mathrm{G})$ *such that for every principal* G*-bundle* $P(X,\mathrm{G})$ *over a* CW*-complex topological space* X*, there exists a unique, up to homotopy, map*

$$\Phi : X \to B\mathrm{G}$$

such that $\Phi^*(E\mathrm{G}(B\mathrm{G},\mathrm{G})) \simeq P(X,\mathrm{G})$*. That is, the classes of isomorphisms of* G*-bundle over a* CW*-complex topological space* X *is classified by a homotopy class of maps from* X *to* $B\mathrm{G}$*.*

$$[X, B\mathrm{G}] \simeq \{\text{Isomorphism classes of principal G-bundles over } X\}.$$

The space $B\mathrm{G}$ *is called the classifying space while the map* Φ *is called the classifying map.*

The universal bundle exists for every topological group G, and it is unique up to homotopy equivalence; moreover, the total space $E\mathrm{G}$ is contractible. The case of complex vector bundles of rank m is recovered looking at the principal bundles $P(X, \mathrm{GL}(m,\mathbb{C}))$; therefore we have

$$B\mathrm{GL}(m,\mathbb{C}) \simeq \mathrm{Gr}_{m,\infty}(S_\infty^\vee). \tag{E.5}$$

However, if $\mathrm{H} \xrightarrow{\iota} \mathrm{G}$ in such a way that G contracts on H, for example $\mathrm{H} = \mathrm{U}(m)$ and $\mathrm{G} = \mathrm{GL}(m,\mathbb{C})$, then $B\mathrm{H} \simeq_h B\mathrm{G}$ and so $H^\bullet(B\mathrm{G}) \simeq H^\bullet(B\mathrm{H})$. Hence, to classify complex vector bundles it is enough to study the cohomology ring $H^\bullet(B\mathrm{U}(m);\mathbb{Z}) \simeq H^\bullet(\mathrm{Gr}_{m,\infty}(S_\infty^\vee);\mathbb{Z})$. This can be computed using the contractility property of $E\mathrm{U}(m)$ and Leray-Serre spectral sequence in the simple case since $\mathrm{U}(m)$ is connected and so, by the long exact sequence in homotopy, $B\mathrm{U}(m)$ is simply connected. The result is the following.

Theorem E.0.4 (Cohomology ring of the infinite Grassmannian)**.** *Let* d_{2i} *be the differential at page* $2i$ *of the Leray-Serre spectral sequence and let* $t_1, ..., t_{2m-1}$ *be the generator of the cohomology ring of* $\mathrm{U}(m)$*,*

$$H^\bullet(\mathrm{U}(m);\mathbb{Z}) = \mathbb{Z}[t_1, ..., t_{2m-1}]/(t_1^2, ..., t_{2m-1}^2), \qquad t_i \in H^i(\mathrm{U}(m);\mathbb{Z}).$$

Then

$$H^\bullet(B\mathrm{U}(m);\mathbb{Z}) = \mathbb{Z}[c_1, ..., c_m];$$

where $c_i = d_{2i}(t_{2i-1}) \in H^{2i}(B\mathrm{U}(m);\mathbb{Z})$*.*

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